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Abstract

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ON THE BOUNDEDNESS IN L_p OF A SINGULAR OPERATOR WITH CAUCHY KERNEL ALONG A CURVE OF BOUNDED ROTATION

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1. Let Γ be an arbitrary Jordan rectifiable curve in the plane $z = x + iy$, and let $f(t)$, $t \in \Gamma$, be an arbitrary function from the space $L_p(\Gamma)$, $p \geqslant 1$. Consider the Cauchy-type integral

$$Kf(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(t)}{t - z}\, dt, \tag{1}$$

understood in the sense of the principal value if $z \in \Gamma$. In the case where Γ is the unit circle, the integral (1), for $z \in \Gamma$ and $p > 1$, defines a continuous operator in the space $L_p(\Gamma)$ (M. Riesz' s theorem). This result was generalized by B. V. Khvedelidze (1947) to the case of Lyapunov curves (see, for example, ([1])).

Let s be the variable arc length of the curve Γ; $t = t(s)$, $0 \leq s \leq S$, the equation of Γ; and let $\gamma(t_1, t_2)$ be the length of the shorter of the two arcs joining the points t_1 and t_2. Suppose, furthermore, that

$$|t_1 - t_2| / \gamma(t_1, t_2) \geq c > 0. \tag{2}$$

In the work of A. G. Dzhvarsheishvili ([2]), M. Riesz' s theorem was generalized to curves Γ satisfying, in addition to condition (2), also a condition of the form $\|t'(s+h) - t'(s)\|_{L_r} \leq C|h|^{\alpha}$. In a recent paper of E. G. Gordadze ([3]) it was proved that M. Riesz' s theorem holds in the case where Γ has no points of cusp, satisfies condition (2), and consists of a finite number of arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, with the singular operator K bounded in each of the spaces $L_p(\Gamma_i)$. Therefore, when convenient, we may restrict ourselves to considering only closed or only open curves.

2. A rectifiable curve Γ has, by definition, bounded rotation if the angle θ between the tangent to it and a fixed direction can be defined so that the function

$\theta(s)$ has bounded variation on the interval $[0, S]$. Curves of bounded rotation include, in particular, all convex curves. Domains whose boundaries have bounded rotation were introduced by I. Radon in connection with problems in the theory of logarithmic potential ([4]). It can be shown that curves of bounded rotation without points of cusp (the maximal jump of the function $\theta(s)$ modulo less than π) satisfy condition (2).

Obviously, it suffices to assume that the density f of the integral (1) is real. Separating the real and imaginary parts, for $z = t_0 \in \Gamma$ we obtain two real operators

$$Pf(t_0) = \frac{1}{2\pi} \int\limits_{\Gamma} f(t)\, ds\, \arg[t(s) - t_0], \qquad t_0 \in \Gamma; \tag{3}$$

$$Qf(t_0) = -\frac{1}{2\pi} \int\limits_{\Gamma} f(t)\, ds\, \ln|t(s) - t_0|, \qquad t_0 \in \Gamma. \tag{4}$$

It was proved by Radon ([4]) that the operator (3) is bounded in the space of continuous functions if the curve Γ has bounded rotation.

The properties of this operator in the spaces $L_p(\Gamma)$ are described by the following proposition.

Theorem 1. *If the curve Γ has bounded rotation and has no cusps, then the operator (3) is bounded in the space $L_p(\Gamma)$ for $p > 1$. If, in addition, Γ has no corner points (i.e., $\theta(s)$ is continuous), then the operator (3) is completely continuous in $L_p(\Gamma)$ for $p > 1$.*

Proof will be given for an open curve Γ. The function $t = t(s) = x(s) + iy(s)$ is absolutely continuous. If we also take into account that almost everywhere $x'(s) = \cos\theta(s)$, $y' = \sin\theta(s)$, and denote by σ the arc length corresponding to the point t_0, then, for $s \neq \sigma$, the kernel of the operator (3) can be represented in the form

$$d_s \arg[t(s) - t(\sigma)] = \frac{y'(s)[x(s) - x(\sigma)] - x'(s)[y(s) - y(\sigma)]}{[x(s) - x(\sigma)]^2 + [y(s) - y(\sigma)]^2} =$$

$$= \int_{\sigma}^{s} \sin[\theta(s) - \theta(\rho)]\, d\rho \Big/ |t(s) - t(\sigma)|^2. \tag{5}$$

The numerator of the last fraction, taken in absolute value, admits the upper estimate

$$|s - \sigma| \sup_{\sigma < \rho < s} |\theta(s) - \theta(\rho)|.$$

Taking into account estimate (2) and representing the function $\theta(s)$ as the difference of nondecreasing functions $\theta_1(s)$, $\theta_2(s)$, we obtain

$$\left|d_s \arg[t(s)-t(\sigma)]\right| \le \frac{1}{c^2}\left[\frac{\theta_1(s)-\theta_1(\sigma)}{s-\sigma}+\frac{\theta_2(s)-\theta_2(\sigma)}{s-\sigma}\right]. \tag{6}$$

The first assertion of Theorem 1 now follows from the fact that $\theta_1(s)$, $\theta_2(s)$ are bounded on the interval $[0, S]$, and the singular integral along the real axis with kernel $(s-\sigma)^{-1}$ is a continuous operator in L_p for $p > 1$.

In the work of S. G. Mikhlin ([5]) it was proved that an operator of the form

$$\int_a^b \frac{g(s)-g(\sigma)}{s-\sigma} f(\sigma)\, d\sigma, \qquad |a|, |b| < +\infty,$$

is completely continuous in $L_p(a, b)$ if the function $g(s)$ is continuous. Hence, from inequality (6) follows the second assertion of Theorem 1 (see ([6]), p. 97).

Remark. Theorem 1 is also valid when the angle of inclination of the tangent is representable in the form $\theta_0(s) + \theta(s)$, where $\theta_0(s)$ is periodic with period S and satisfies a Hölder condition with exponent $0 < \alpha \le 1$; $\theta(s)$ is the function of bounded variation considered above.

3. Suppose that Γ is closed and the origin of coordinates lies inside the (finite) domain G bounded by Γ. Let $\varphi(t)$ denote the principal branch of the function $\arg t$, $t \in \Gamma$. We shall prove that, if Γ has bounded rotation and is free of cusps, then there exist a number $0 < \delta < \pi/2$ and a function $\Phi_0(z)$, analytic in G and continuous in $G + \Gamma$, such that

$$\sup_{t\in\Gamma} |\theta(t) - \varphi(t) + \operatorname{Re}\Phi_0(t)| \le \delta < \pi/2. \tag{7}$$

Indeed, let $z = w(\zeta)$ map conformally the unit disk $|\zeta| \le 1$ onto G. Since $\theta[w(e^{is})]$ has bounded variation on $[0, 2\pi]$ and all its jumps are, in modulus, less than π, $\varphi[w(e^{is})]$ is continuous on $[0, 2\pi]$, and the difference $\theta[w(e^{is})] - \varphi[w(e^{is})]$ may be regarded as discontinuous at the points $s = 0 \pmod{2\pi}$, there exists a function $\mu(s)$, satisfying a Lipschitz condition on $[0, 2\pi]$, 2π-periodic and such that

$$|\theta[w(e^{is})] - \varphi[w(e^{is})] + \mu(s)| \le \delta < \pi/2.$$

If now the Schwarz integral with density μ, using the inverse of $w(\zeta)$, is expressed in terms of the variable z, we obtain the required function $\Phi_0(z)$. Put $\Phi(z) = \exp i\Phi_0(z)$.

Theorem 2. *If the curve Γ has bounded rotation and has no cusps, then the singular integral operator K, defined by formula (1) for $z \in \Gamma$, is bounded in the space $L_p(\Gamma)$ for $p > 1$.*

Proof. We shall carry out the proof for the case of a closed curve Γ. Let f belong to the set N, everywhere dense in $L_p(\Gamma)$, of functions satisfying a

Lipschitz condition, and let $p = 2n$, $n = 1, 2, \ldots$. Then the integrals (1) (for $z = t_0$), (3), (4) exist at every point $t_0 \in \Gamma$, and the Cauchy-type integral (1) is a continuous function in $G + \Gamma$; moreover the real and imaginary parts of the angular boundary values from inside G are expressed respectively by the formulas $u = \frac{1}{2} f + Pf$, $v = Qf$ (see (7), pp. 197, 188). Consider the integral

$$I = \frac{1}{2\pi i} \int_\Gamma (iv)^p \frac{\Phi(t)}{t}\, dt. \tag{8}$$

Taking into account that

$$|I| \leq \frac{1}{2\pi} \int_\Gamma |(iv)^p - (u+iv)^p| \left|\frac{\Phi(t)}{t}\right| ds + \left|\frac{1}{2\pi i} \int_\Gamma (u+iv)^p \frac{\Phi(t)}{t}\, dt\right|,$$

and that the function $\Phi(t)$ is bounded in $G + \Gamma$, and

$$|(u+iv)^p - (iv)^p| \leq p 2^{(p-1)/2} (|u| + |u||v|^{p-1}),$$

we obtain

$$|I| \leq M_1 (\|u\|_{L_p} \|v\|_{L_p}^{p-1} + \|u\|_{L_p}) + M_2 |Kf(0)|^p, \quad M_1, M_2 = \text{const}.$$

From formula (1) it is easy to obtain the inequality $|Kf(0)| \leq M_3 \|f\|_{L_p}$, $M_3 =$ const, and by Theorem 1 we have $\|u\|_{L_p} \leq M_4 \|f\|_{L_p}$, $M_4 =$ const. Therefore, from the preceding inequality one can obtain the following:

$$|I| \leq \alpha \|v\|_{L_p}^{p-1} \|f\|_{L_p} + \beta \|f\|_{L_p}^p, \tag{9}$$

where α, β are positive constants independent of f. To estimate the integral (8) from below, we use inequality (7):

$$|I| = \left| \int_\Gamma |v|^p \left|\frac{\Phi(t)}{t}\right| \exp i\{\theta - \varphi + \arg\Phi\}\, ds \right| \geq$$

$$\geq \int_\Gamma |v|^p \frac{\exp\{-\operatorname{Im}\Phi_0\}}{|t|} \cos\{\theta - \varphi + \operatorname{Re}\Phi_0\}\, ds \geq \gamma \|v\|_{L_p}^p, \tag{10}$$

where γ is a positive number independent of f. From formulas (9), (10) we obtain

$$\gamma \|v\|_{L_p}^p \leq \alpha \|v\|_{L_p}^{p-1} \|f\|_{L_p} + \beta \|f\|_{L_p}^p. \tag{11}$$

It follows that the ratio $\|v\|_{L_p}/\|f\|_{L_p}$ is bounded above by the unique positive root of the equation $\gamma x^p = \alpha x^{p-1} + \beta$. In other words, the operator Q, and with it also the operator K, defined by formula (1) for $z \in \Gamma$, is bounded on the set N, everywhere dense in $L_p(\Gamma)$. From the latter assertion it follows, further, that the Cauchy-type integral (1) represents a function of the class $E_p(G)$, whatever the function f in $L_p(\Gamma)$ may be (see (8), p. 512). Consequently, it has angular boundary values almost everywhere on Γ, and the preceding considerations can be carried out at once for any function $f \in L_p(\Gamma)$, since functions of the class $E_1(G)$ are representable by a Cauchy integral (see (7), p. 205).

For arbitrary $p \geq 2$, the theorem follows from what has been proved and from the interpolation theorem of M. Riesz (see, for example, (6), p. 34). Finally, the remaining case $1 < p < 2$ is easily considered if one passes to the adjoint operators in the conjugate space $L_{p'}(\Gamma)$, $p' = p/(p-1) > 2$, and refers to the case already analyzed. Theorem 2 is completely proved.

4. The basic idea of the method by which inequality (11) was obtained belongs to M. Riesz (see, for example, (9), p. 149).

Let, for the curve Γ, the angle of inclination of the tangent be represented in the form $\theta_0(s) + \theta(s)$, where the function $\theta_0(s)$ is continuous and periodic with period S, while $\theta(s)$ is the function of bounded variation considered above. If by $u + iv$ one understands the limiting value on Γ of an arbitrary function F of the class $E_p(G)$ satisfying the additional condition $|F(0)| \leq M\|u\|_{L_p}$, $M = \text{const}$, then the preceding arguments lead to an inequality of the form

$$\gamma\|v\|^p_{L_p} \leq \alpha\|v\|^{p-1}_{L_p}\|u\|_{L_p} + \beta\|u\|^p_{L_p}. \tag{11'}$$

It follows from this that the ratio $\|v\|_{L_p}/\|u\|_{L_p}$ is bounded, which in the case of the unit disk is equivalent to M. Riesz' s theorem on the boundedness of the singular operator. In this sense M. Riesz' s theorem was generalized in the paper of V. P. Khavin [8] to the case of piecewise smooth Γ.

Remark. In the study of general regularized boundary-value problems for elliptic systems of linear differential equations, in the work of Ya. B. Lopatinskii [10] integral equations were obtained containing "generalized double-layer potentials" of type (3). It can be shown that the assertions of Theorem 1 remain valid also for these more general potentials.

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Note: Figure translations are in progress. See original paper for figures.