More on exact estimates in Rothe' s method

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Abstract

The main results of the authors' previous note are generalized to a general linear second-order parabolic equation with a sign-definite lower-order coefficient $c(x, t)$ under very general boundary conditions. The presentation is primarily conducted for the case of a one-dimensional problem with continuous coefficients, but the validity of the results is also demonstrated for discontinuous problems, including multidimensional ones.

Using the maximum principle and the method of majorants, the following error estimates for the Rothe method are obtained:

$$|\varepsilon_n| \leq \frac{h}{2} t_{nM} 2,$$

$$|\varepsilon_n| \leq \frac{h}{2c} M_2 [1 - (1 + hc)^{-n}], \quad 0 < c \leq c(x, t),$$

where M_2 is the supremum of the modulus of the second derivative of the exact solution with respect to time t. These estimates are achieved within the class of problems under consideration. The estimates obtained in this work for the solution of the original problem possess a similar property. The error estimates are also valid for certain nonlinear problems.

Bibliography: 4.

Full Text

Preamble

This work, published in 1967 (Vol. III, No. 8), extends the methods developed by V. V. Bobkov and O. A. Liskovets [?]. We consider the numerical solution of the boundary value problem for a linear parabolic equation of the form:

$$L(u) = a(x, t) u_{xx} + b(x, t) u_x - c(x, t) u - u_t = f(x, t)$$

subject to the conditions $a(x,t) \geq a_0 > 0$, $c(x,t) \geq 0$ in the domain $0 < x < 1, 0 < t \leq T$. The initial condition is given by $u(x,0) = \phi(x)$, where $\phi(x) \in C[0,1]$. The boundary conditions are defined as:

$$l^{(0)}(u) = \alpha_0(t)u_x(0,t) - \beta_0(t)u(0,t) = -\psi_0(t)$$
$$l^{(1)}(u) = \alpha_1(t)u_x(1,t) + \beta_1(t)u(1,t) = \psi_1(t)$$

where $\alpha_i(t) \geq 0, \beta_i(t) \geq 0$, and $\alpha_i(t) + \beta_i(t) > 0$.

To solve this problem, we employ a discretization scheme with respect to time. Let $t_n = nh$, where $h > 0$ is the time step and $n = 0, 1, \dots, N = [T/h]$. We denote the approximate solution at the n-th time level as $u_n(x)$. The differential-difference equation is then formulated as:

$$L_n(u_n) = a_n(x)u_n'' + b_n(x)u_n' - c_n(x)u_n - \frac{u_n - u_{n-1}}{h} = f_n(x)$$

with the corresponding boundary conditions $l^{(0)}(u_n) = -\psi_0(t_n)$ and $l^{(1)}(u_n) = \psi_1(t_n)$. The initial state is $u_0(x) = \phi(x)$.

Error Analysis and Convergence

Let $\epsilon_n(x) = u(x,t_n) - u_n(x)$ represent the error at the n-th step. By substituting this into the operator L_n, we obtain the error equation $L_n(\epsilon_n) = r_n(x)$, where $r_n(x)$ is the truncation error. Under the assumption that the solution $u(x,t)$ is sufficiently smooth, specifically $u \in C^2$, the residual is bounded by:

$$|r_n(x)| \leq R = \frac{h}{2}M_2, \quad M_2 = \max|u_{tt}(x,t)|$$

As $h \to 0$, the residual $r_n(x)$ vanishes, suggesting convergence of the scheme. We aim to show that $|\epsilon_n(x)| \leq Q(h)$, where $\lim_{h\to 0} Q(h) = 0$.

To establish the stability and convergence of the method, we utilize a comparison principle. Suppose there exists a function $v_n(x) \in C[0,1] \cap C^2(0,1)$ such that $L_n(v_n) > 0$ in the domain. If the boundary conditions satisfy certain positivity requirements, then the maximum principle ensures that the error remains bounded. Specifically, if $c(x,t) \geq c_0 > 0$, the error satisfies:

$$|\epsilon_n(x)| \leq \frac{hM_2}{2c_0}[1 - (1 + hc_0)^{-n}]$$

This confirms that the approximate solution $u_n(x)$ converges uniformly to the exact solution $u(x,t_n)$ as the step size h decreases.

Generalizations and Nonlinear Cases

The method can be extended to more complex scenarios, including problems with discontinuous coefficients or non-classical boundary conditions. For instance, if the coefficients $a(x,t)$ or $b(x,t)$ have a jump discontinuity at some point $\xi \in (0,1)$, we introduce matching conditions:

$$[u]_{x=\xi} = 0, \quad [p(t)u_x]_{x=\xi} = g(t)$$

where $p(t) > 0$. The discrete operator L_n is applied separately in the sub-intervals $(0,\xi)$ and $(\xi,1)$, and the matching conditions are discretized accordingly.

Furthermore, the analysis extends to quasi-linear equations of the form $u_t = a(x,t,u,u_x)u_{xx} + f(x,t,u,u_x)$. Under the assumption of ellipticity ($a > 0$) and certain monotonicity conditions on the nonlinear terms (e.g., $\partial f/\partial u \geq c > 0$), the convergence of the time-discretization method can still be proven using similar comparison theorems.

Multidimensional Extension

The approach is also applicable to multidimensional parabolic equations in a domain D:

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t)\frac{\partial u}{\partial x_i} - c(x,t)u - u_t = f(x,t)$$

where the operator is uniformly elliptic. By discretizing the time derivative, we reduce the problem to a sequence of elliptic boundary value problems at each time step t_n. The convergence results hold provided the domain boundary and the coefficients are sufficiently smooth to guarantee the existence of solutions to the resulting elliptic problems.

References

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Note: Figure translations are in progress. See original paper for figures.