ON ANGULAR BOUNDARY VALUES OF QUASICONFORMAL MAPPINGS OF A BALL

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.53

MATHEMATICS

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ON ANGULAR BOUNDARY VALUES OF QUASICONFORMAL MAPPINGS OF A BALL

(Presented by Academician M. A. Lavrent' ev, 17 II 1967)

We shall use the following notation: R^n is Euclidean space of dimension n; B is an open ball in R^n; S is the boundary of B; $\overline{B} = B \cup S$; $\text{mes}_\alpha E$ is the α-dimensional Hausdorff measure of the set E; $\overline{\text{cap}}_\alpha E$ is the outer α-capacity of the set E; $y = y(x)$ is a mapping of B into R^n; A is the set of points on S at which the mapping $y = y(x)$ has angular boundary values; $M(\Gamma)$ is the modulus of the family of curves Γ.

Theorem. If $y = y(x)$ is a quasiconformal mapping of the ball $B \subset R^n$, then

$$\overline{\text{cap}}_0(S \setminus A) = 0 \quad \text{for } n = 2$$

$$\overline{\text{cap}}_\alpha(S \setminus A) = 0 \quad \text{for } n > 2 \text{ and any } \alpha > 0.$$

First of all we make several remarks that will be used in the proof.

1°. Let Γ' be the family of all possible curves lying in $\overline{B}$ and ending at points of some set $E \subset S$, and let Γ'' be the family of all curves in R^n intersecting E; then $M(\Gamma'') = 2M(\Gamma')$.

By virtue of the conformal invariance of the modulus, it is enough to prove this assertion for the case when Γ' is the family of curves from the half-space $x_n \geq 0$, E is a set on the hyperplane $x_n = 0$, and Γ'' is the family of all curves in R^n having common points with E. But in this case, in computing $M(\Gamma'')$, one may restrict oneself only to admissible metrics taking equal values at the points $x, \overline{x}$, symmetric with respect to the hyperplane $x_n = 0$. Indeed, if $\rho(x)$ is an admissible function for Γ'', then $\rho(\overline{x})$, obviously, is also admissible for Γ'', and consequently so is the function $\tilde{\rho}(x) = \frac{1}{2}[\rho(x) + \rho(\overline{x})]$, and

$$\int_{R^n} \tilde{\rho}^n(x)\, dx = \int_{R^n} \left[\frac{\rho(x) + \rho(\overline{x})}{2}\right]^n dx \leq$$

$$\leq \frac{1}{2}\int_{R^n} \rho(x)\,dx + \frac{1}{2}\int_{R^n} \rho(\overline{x})\,dx = \int_{R^n} \rho^n(x)\,dx.$$

Since

$$\int_{R^n} \tilde{\rho}^n(x)\,dx = 2\int_{x_n \geq 0} \tilde{\rho}^n(x)\,dx,$$

it follows that

$$2M(\Gamma') \leq M(\Gamma'').$$

On the other hand, if $\rho'(x)$ is a function in $x_n \geq 0$ admissible for Γ', then the function

$$\rho(x) = \begin{cases} \rho'(x), & \text{for } x_n \geq 0, \\ \rho'(\overline{x}), & \text{for } x_n < 0 \end{cases}$$

admissible for Γ''. Consequently, $M(\Gamma'') \leq 2M(\Gamma')$, and assertion 1° is proved.

2°. Let E be a set in R^n whose full measure is equal to zero; then the family Γ of curves in R^n, each of which has intersection with E of nonzero linear measure, is exceptional, i.e. $M(\Gamma) = 0$.

For the proof it suffices to consider the function

$$\rho(x) = \begin{cases} +\infty, & \text{if } x \in E, \\ 0, & \text{if } x \notin E, \end{cases} \qquad \int_{R^n} \rho^n(x)\,dx = 0,$$

admissible for Γ.

If $y = y(x)$ is a quasiconformal mapping of the ball B, and A' is the set of points on S such that for any point $x \in A'$ there exists a path $\gamma_x \subset B$, going to x, along which the mapping has a limit, then, as Gehring showed ([2]), $A' = A$.

Now let γ be a curve in $\overline{B}$, ending at some point $x_0 \in S$ and having intersection with S of linear measure zero (in the case when γ has common points with S at all, apart from the point x_0). We shall say that the mapping $y = y(x)$ has at x_0 a generalized limit along γ, if it has at x_0 a limit over the set $\gamma \cap B$. Denote by A'' the totality of those points on S for each of which one can indicate a path $\gamma \subset \overline{B}$, going to this point, along which the mapping has a generalized limit.

Introducing into Gehring' s arguments ([2]) the obvious changes, it is easy to verify that

3°. $A = A' = A''$.

Let Γ be a family of curves in $\overline{B}$ such that, if $\gamma \in \Gamma$, then either γ has no common points with S, or $\mathrm{mes}_1(\gamma \cap S) = 0$. The set $\dot{\gamma} = \gamma \setminus (\gamma \cap S) = \gamma \cap B$ will be called the generalized curve. We shall say that $\dot{\gamma}$ goes to the point x_0 (or ends at x_0) if the same was true for γ.

If now $\dot{\Gamma}$ is the family of generalized curves obtained from the family Γ, then, obviously:

4°. $M(\dot{\Gamma}) = M(\Gamma)$.

Finally, just as for a family of ordinary curves, for a family of generalized curves under a Q-quasiconformal mapping we have:

5°. $Q^{-n}M(\dot{\Gamma}_*) \leq M(\dot{\Gamma}) \leq Q^{nM}(\dot{\Gamma}_*)$, where $\dot{\Gamma}_*$ is the image of the family $\dot{\Gamma}$.

We now prove the theorem formulated above.

Let $\dot{\Gamma}$ be the family of all generalized curves in B ending at points of the set $S \setminus A''$. Then for the image $\dot{\gamma}_*$ of any generalized curve $\dot{\gamma} \in \dot{\Gamma}$ we have $\mathrm{mes}_1(\dot{\gamma}_*) = +\infty$, whence it follows ([4]) that $M(\dot{\Gamma}_*) = 0$. By virtue of 5° we have $M(\dot{\Gamma}) = 0$, and from 4° we conclude that $M(\Gamma) = 0$. But then from 2° and 1° it follows that the family of all curves in R^n having common points with $S \setminus A''$ is also exceptional. On the basis of Pfluger' s theorem 7 ([3]) we conclude that

$$\overline{\mathrm{cap}}_0(S \setminus A'') = 0 \qquad \text{for } n = 2$$

$$\overline{\mathrm{cap}}_\alpha(S \setminus A'') = 0 \qquad \text{for } n > 2 \text{ for any } \alpha > 0.$$

To complete the proof it remains to refer to remark 3°. Thus, for $n = 2$ we obtain the known theorem of Beurling ([1]). In the case $n > 2$, however, the assertion proved is equivalent to the assertion that the set $(S \setminus A)$ has metric dimension zero. Consequently, there still remains open the question: is Beurling' s theorem true in its full scope for spaces of dimension $n > 2$?

In conclusion I note that, as became clear at the International Mathematical Congress in Moscow in 1966, the same result was obtained by the American mathematician T. Gehring. However, our proofs turned out to be so different that we had to abandon the joint publication that would be natural in such a situation.

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Received
19 I 1967

CITED LITERATURE

[1] A. Beurling, Acta Math., 72, No. 1-2 (1940). [2] F. W. Gehring, Ann. Acad. Sci. Fenn., Ser. AI, No. 336/11 (1963). [3] B. Fuglede, Acta Math., 98, No. 3-4 (1957). [4] J. Väisälä, Ann. Acad. Sci. Fenn., Ser. AI, No. 298 (1961).

Note: Figure translations are in progress. See original paper for figures.