THERMODYNAMIC RELATIONS ON THE SURFACE OF A STRONG DISCONTINUITY IN AN ELASTIC MEDIUM UNDER FINITE DEFORMATIONS

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Abstract

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THERMODYNAMIC RELATIONS ON THE SURFACE OF A STRONG DISCONTINUITY IN AN ELASTIC MEDIUM UNDER FINITE DEFORMATIONS

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Consider a thermodynamically isolated system consisting of an elastic material. By definition (^{1}), for an elastic medium in a neighborhood of the unstressed natural state there exists a one-to-one relation between the Eulerian stress tensor σ_{ij} and the Almansi finite-strain tensor e_{ij}.

Suppose that in the space filled by the elastic medium there moves a surface of strong discontinuity Σ (a velocity-discontinuity wave). Let us write the conditions of dynamic compatibility of two continuous motions of the medium occurring on different sides of the discontinuity surface. The conditions expressing the laws of conservation of mass and momentum have the form (^{2}):

$$[\rho\theta] = 0, \qquad [\rho\theta v_i] = -[\sigma_{ij}]\nu_j. \tag{1}$$

Summation over repeated indices is assumed here and below; ρ is the density; v_i are the components of the particle velocity; ν_j is the unit normal vector to the discontinuity surface, directed toward the side of its propagation; $\theta = N - v_i\nu_i$ is the normal propagation velocity of the surface Σ. The symbol $[\,]$ denotes the jump of a function, where

$$[a] = a_1 - a_2, \qquad [ab] = [a]b_2 + [b]a_2 + [a][b],$$

where the indices 1 and 2 refer to parameters calculated, respectively, on the rear and front sides of the discontinuity surface.

In addition to relations (1), on the discontinuity surface an energetic compatibility condition must be satisfied, expressing the law of conservation of energy. For an arbitrary volume τ, bounded by a surface σ, we shall have (^{3,4})

$$\frac{d}{dt}\int_{(\tau)}\left(\rho\frac{v_i v_i}{2}+\rho U\right)d\tau=\int_{(\sigma)}\sigma_{ij}v_i\nu_j\,d\sigma. \tag{2}$$

The internal energy U per unit mass may be represented as the sum of reversible (potential energy of deformation) and irreversible parts,

$$U=\varepsilon+TS. \tag{3}$$

It is assumed that outside the discontinuity surface the processes are reversible, so that the entropy S can change only in passing through the surface Σ, on which deformation occurs with dissociation of mechanical energy.

Letting in (2) the integration volume τ tend to an element of the discontinuity surface $\Delta\Sigma$ contained in it, we obtain, taking (3) into account,

$$\left[\rho\theta\left(\frac{v_i v_i}{2}+\varepsilon\right)+[\sigma_{ij}v_i]\nu_j\right]=-[\rho\theta TS]\le 0. \tag{4}$$

The inequality sign is placed here because, according to the second law of thermodynamics ([5]), the entropy of a thermally insulated system can only incre –

increase. The energy condition (4), written in Eulerian variables, is analogous to the thermodynamic relations obtained in ([6],[7]) in Lagrangian variables.

Suppose that, ahead of the discontinuity surface, the material is in a natural undeformed state at rest. Then relations (1) and (4) may be written in the form

$$[\rho\theta]=0,\qquad \rho\theta[v_i]=-[\sigma_{ij}]v_j, \tag{5}$$

$$\rho_1\theta_1^2[E]\le\frac{1}{2}[\sigma_{ij}][\sigma_{ik}]v_j v_k, \tag{6}$$

where E is the potential deformation energy per unit volume in the instantaneous state.

In addition to the dynamic conditions (5)–(6), the kinematic compatibility conditions must hold on the discontinuity surface,

$$[u_{i,j}]=\omega_i v_j,\qquad [\partial u_i/\partial t]=-N\omega_i, \tag{7}$$

where u_i is the displacement vector; $u_{i,j}$ are the displacement gradients; ω_i is the wave vector; N is the velocity of propagation of the wave; the comma denotes differentiation with respect to the spatial coordinates.

Let us apply the kinematic and dynamic compatibility conditions to study the propagation of discontinuity surfaces in the geometrically nonlinear Almansi medium, whose constitutive equation has the form

$$\sigma_{ij} = \lambda \sigma_{ij} e_{kk} + 2\mu e_{ij}, \qquad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}), \tag{8}$$

where λ, μ are positive constants.

From (5), (7), (8), after transformations that are omitted here, one can obtain the relations

$$\lambda \cos\varphi(\cos\varphi - \tfrac{1}{2}\omega) + \mu(1 + \cos^2\varphi - \omega\cos\varphi) = \rho_1\theta_1^2,$$

$$\lambda(\cos\varphi - \tfrac{1}{2}\omega) + \mu(2\cos\varphi - \omega) = \rho_1\theta_1^2\cos\varphi, \tag{9}$$

$$\rho_1 = \rho_2(1 - \omega\cos\varphi), \qquad \omega_i v_i = \omega\cos\varphi, \qquad \omega = |(\omega_k\omega_k)^{1/2}|.$$

Hence, for the velocity θ_1, we obtain three values:

$$\theta_1^2 = \frac{\lambda + 2\mu}{\rho_1}\left(1 \mp \frac{1}{2}\omega\right), \qquad \cos\varphi = \pm 1; \tag{10}$$

$$\theta_1^2 = \frac{\mu}{\rho_1}, \qquad \cos = \frac{\lambda + 2\mu}{2(\lambda + \mu)}\,\omega = \cos\varphi_*. \tag{11}$$

In formulas (10) the upper sign corresponds to a longitudinal rarefaction wave, and the lower sign to a longitudinal compression wave. Relations (11) occur for oblique shear waves, when the wave vector is inclined to the normal vector at the angle φ_*.

Let us apply the energy condition (6). To compute E we shall use the known (1) differential relation of Noll, connecting the components of the Eulerian stress tensor σ_{ij} with the components of the finite-strain Almansi tensor:

$$\sigma_{ij} = \frac{\rho}{\rho_2}\frac{\partial E}{\partial e_{ik}}(\delta_{kj} - 2e_{kj}), \tag{12}$$

where the left-hand side must be replaced by the defining relation (8). In general form, equation (12) cannot be solved differentially with respect to E. If, however, it is taken into account that, in passing through the discontinuity surface, the displacement gradients have the specific structure (7), such a solution becomes possible.

Namely, we shall assume that at any moment of time, in any neighborhood of the undeformed state, the displacement gradients can be represented in the form

$$u_{i,j} = w_i v_j, \tag{13}$$

where ν_i are the direction cosines of the normal to the surface of discontinuity, and w_i are functions characterizing the "intensity" of the displacement gradient. On the surface of discontinuity w_i varies from zero to ω_i.

Taking (13) into account, for the invariants of the strain tensor we obtain

$$\begin{aligned} I_1 &= e_{kk} = \frac{1}{2}(2w\cos\varphi - w^2), \\ I_2 &= \frac{1}{2}\delta^{ik}_{jl} e_{ij} e_{kl} = \frac{1}{4}(w^2\cos^2\varphi - w^2), \\ I_3 &= \frac{1}{6}\delta^{ijk}_{lmn} e_{il} e_{jm} e_{kn} = 0. \end{aligned} \tag{14}$$

Here φ is the angle between the vectors w_i, ν_i; $\sqrt{w_k w_k} = w$. It follows from (14) that E does not depend on the third invariant, and if $\cos\varphi = \pm 1$, it also does not depend on the second. Taking this remark into account, we write equation (12) in the form

$$\sigma_{ij} = \frac{\rho}{\rho_2}\left(\frac{\partial E}{\partial I_1}\frac{\partial I_1}{\partial e_{ik}} + \frac{\partial E}{\partial I_2}\frac{\partial I_2}{\partial e_{ik}}\right)(\delta_{kj} - 2e_{kj}), \tag{15}$$

where, for $\cos\varphi = \pm 1$, the term containing $\partial E/\partial I_2$ must be put identically equal to zero. For ρ/ρ_2 we have

$$\rho/\rho_2 = \sqrt{1 - 2I_1 + 4I_2 - 8I_3} = 1 - w\cos\varphi. \tag{16}$$

Multiply equation (15) by $w_i \nu_j$ and sum over repeated indices. Using (8), (13), (14), (16), we obtain

$$\begin{gathered} \frac{1}{2}\lambda\cos\varphi(2w\cos\varphi - w^2) + \mu(w + w\cos^2\varphi - w^2\cos\varphi) \\ = (1-w\cos\varphi)\left\{\frac{\partial E}{\partial I_1}(\cos\varphi - w - w\cos^2\varphi + w^2\cos\varphi) + \frac{1}{2}\frac{\partial E}{\partial I_2}\sin^2\varphi(w^2\cos\varphi - w)\right\}. \end{gathered} \tag{17}$$

Taking $\cos\varphi$ equal to ± 1 or to $\cos\varphi_*$, from (17) we obtain an ordinary differential equation determining the dependence of E on w in the three cases of interest to us, (10) and (11):

$$\frac{dE}{dw^2} = \frac{\lambda + 2\mu}{2} \frac{2w \mp w^2}{(1 \mp w)^2}, \qquad \cos\varphi = \pm 1; \tag{18}$$

$$\frac{1}{2}\lambda a(2aw^2 - w^2) + \mu(1 + a^2w^2 - aw^2)$$
$$= (1 - aw^2)\frac{dE}{dw^2}\left\{\frac{2(a - 1 - a^2w^2 + aw^2)}{2a - 1} + 2(1 - a^2w^2)\frac{aw^2 - 1}{2a^2w^2 - 1}\right\}, \tag{19}$$

$$\cos\varphi = aw, \qquad a = \frac{\lambda + 2\mu}{2(\lambda + \mu)}.$$

Integrating (18) under the initial conditions $E = 0$ at $w = 0$, we obtain

$$E = \frac{\lambda + 2\mu}{2} \frac{w^2}{1 \mp w}, \qquad \cos\varphi = \pm 1. \tag{20}$$

Relation (6), with the use of (7), (8), (10), (13), (20), leads to the inequality

$$1 \le (1 \mp \omega)\left(1 \mp \frac{1}{2}\omega\right), \qquad \cos\varphi = \pm 1, \tag{21}$$

which is satisfied only for compression waves ($\cos\varphi = -1$).

We integrate equation (19) under the assumption $w^2 \ll 1$:

$$E = \frac{\mu^2}{2\mu - \lambda} w^2.$$

Substituting this into (6) and taking into account that

$$[\sigma_{ij}][\sigma_{ik}]\nu_j\nu_k \approx \mu^2\omega^2, \qquad \rho_1\theta_1^2 = \mu,$$

we obtain

$$\lambda \le 0,$$

which contradicts the original assumptions of the model.

Thus, in an Almansi medium the only shock wave whose existence would not contradict the second law of thermodynamics is a longitudinal compression shock wave.

We note that the results for infinitely small deformations can be obtained in the limit as $\omega \to 0$. Then the process on the discontinuity surface is essentially reversible, and in (6) the limiting equality is satisfied for all three types of waves; moreover, the oblique-shear wave degenerates into a transverse wave ($\varphi = \pi/2$).

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Note: Figure translations are in progress. See original paper for figures.