AVERAGING IN SYSTEMS OF DIFFERENTIAL EQUATIONS NOT SOLVED WITH RESPECT TO THE DERIVATIVE

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Abstract

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AVERAGING IN SYSTEMS OF DIFFERENTIAL EQUATIONS NOT SOLVED WITH RESPECT TO THE DERIVATIVE

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Numerous works concerning the averaging method in systems of differential equations are devoted to the consideration of this method in systems of differential equations solved with respect to the derivative. For a detailed bibliography of the works mentioned, see ([1−3]).

The present note is devoted to extending the averaging method to systems of ordinary differential equations not solved with respect to the derivative.

I. Consider the system

$$\dot{x} = \varepsilon X(t, x, \dot{x}), \tag{1}$$

where x, X are n-dimensional vectors, $\varepsilon > 0$ is a small parameter.

Suppose that the limit exists

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, x, \dot{x})\, dt = X_0(x, \dot{x}). \tag{2}$$

Along with system (1), consider the averaged system

$$\dot{\xi} = \varepsilon X_0(\xi, \dot{\xi}). \tag{3}$$

Concerning the closeness of the solutions of equations (1) and (3), the following can be proved.

Theorem. Let the function $X(t, x, y)$, continuous in all variables, satisfy the conditions:

1. For some domain $D \subset E_{2n}(x, y)$, one can indicate constants M and λ such that, for all $t \geq 0$ and any points (x, y), (x', y'), (x'', y'') from this domain, the inequalities

$$|X(t,x,y)| \leq M,$$

$$|X(t,x',y') - X(t,x'',y'')| \leq \lambda\{|x'-x''| + |y'-y''|\}.$$

2. Uniformly with respect to x and y in the domain D, the limit (2) exists and

$$|\partial X_0/\partial x| \leq P, \quad |\partial X_0/\partial y| \leq Q$$

(P and Q are constants).

Suppose, furthermore, that $\xi = \xi(t)$ is a solution of equation (3), defined on the interval $0 \leq t < \infty$ and having the property that the set $\{\xi(t), \dot{\xi}(t)\}$, as t varies in the interval $0 \leq t < \infty$, lies in the domain D together with some ρ-neighborhood.

Then to every arbitrarily small positive η and arbitrarily large $L > 0$ one can put in correspondence an ε_0 such that, for $0 < \varepsilon < \varepsilon_0$, on the interval $0 < t < L\varepsilon^{-1}$ the inequality

$$|\xi(t) - x(t)| < \eta, \tag{4}$$

will hold, where $x(t)$* is a solution of equation (1) coinciding with $\xi(t)$ at $t = 0$.

* It is assumed that there exists a unique solution of equation (1) satisfying the initial condition $x(0) = \xi(0)$ and defined for all $t \geq 0$.

Proof. Introduce the function

$$u(t,x,y) = \int_D \Delta_a(x-x',\, y-y') \times$$

$$\times \left\{ \int_0^t [X(\tau,x',y') - X_0(x',y')]\, d\tau \right\} dx'\, dy',$$

where

$$\Delta_a(x,y) =$$

$$= \begin{cases} A(1-|x|^2/a^2)^2(1-|y|^2/a^2)^2, & |x| > a,\ |y| > a, \\ 0, & |x| \leq a,\ |y| \leq a, \end{cases} \qquad \int_{E_{2n}} \Delta_a(x,y)\, dx\, dy = 1.$$

Using the conditions of the theorem, it is not hard to obtain the following estimates for the function $u(t,x,y)$ and its derivatives:

$$|u| \leq t\alpha(t), \qquad |\partial u/\partial x| \leq It\alpha(t), \qquad |\partial u/\partial y| \leq It\alpha(t),$$

$$|\partial u/\partial t| \leq 2M, \qquad |\partial u/\partial t - X(t,x,y) + X_0(x,y)| \leq 4\lambda a. \tag{5}$$

Here $\alpha(t)$ is a monotonically decreasing function,

$$I = \max\left\{\int_{E_{2n}} \left|\frac{\partial \Delta_a}{\partial x}\right| dx\, dy, \quad \int_{E_{2n}} \left|\frac{\partial \Delta_a}{\partial y}\right| dx\, dy\right\}.$$

(We note that the last inequality in (5) is valid for those points $(x, y) \in D$ whose a-neighborhood belongs to D.) For $a < \rho$ we find that

$$\bar{x}(t) = \xi(t) + \varepsilon u(t, \xi(t), \dot{\xi}(t))$$

belongs to D on the interval $0 < t < L\varepsilon^{-1}$.

Introduce the function $R(t)$, setting $R(t) = \dot{\bar{x}} - \varepsilon X(t, \bar{x}, \dot{\bar{x}})$. Carrying out the calculations, we find

$$R = \varepsilon\left\{\frac{\partial u}{\partial t} - X(t,\xi,\dot{\xi}) + X_0(\xi,\dot{\xi})\right\} + \varepsilon^2 \frac{\partial u}{\partial \xi} X_0(\xi,\dot{\xi}) +$$

$$+\varepsilon^3 \frac{\partial u}{\partial \dot{\xi}} \frac{\partial X_0}{\partial \xi}\left(1 - \varepsilon \frac{\partial X_0}{\partial \dot{\xi}}\right)^{-1} X_0 + \varepsilon\{X(t,\xi,\dot{\xi}) - X(t,\xi,\dot{\xi} + \varepsilon u)\} +$$

$$+\varepsilon\{X(t,\xi,\dot{\xi} + \varepsilon \dot{u}) - X(t, \xi + \varepsilon u, \dot{\xi} + \varepsilon \dot{u})\}.$$

Using the estimates (5), we obtain

$$|R(t)| \leq 4\varepsilon\lambda a + (2\lambda + c_1 Ia)\varepsilon^2 t f(t) + c_2 Ia\varepsilon^3 t f(t) +$$

$$+\varepsilon^2\lambda[2M + \varepsilon c_3 t f(t) + \varepsilon^2 Iac_4 t f(t)] \tag{6}$$

(c_1, c_2, c_3, c_4 are constants).

On the other hand, on the interval $0 < t < t^*$, $t^* \leq L\varepsilon^{-1}$, on which $x(t) \in D$, we have

$$\left|\frac{d}{dt}(\bar{x} - x)\right| \leq \frac{\varepsilon\lambda}{1 - \varepsilon\lambda}|\bar{x} - x| + \frac{|R(t)|}{1 - \varepsilon\lambda}, \qquad (\bar{x} - x)_{t=0} = 0.$$

It follows that

$$|\bar{x} - x| \leq \int_0^t e^{\varepsilon\lambda(1-\varepsilon\lambda)^{-1}(t-\tau)} |R(\tau)| \, d\tau. \tag{7}$$

It is not hard to show that, by disposing of the parameters a and ε, one can ensure that on the interval $0 < t < L\varepsilon^{-1}$ the integral standing on the right-hand side of inequality (7) is less than $\frac{1}{2}\min(\rho, \eta)$. Therefore, on the interval $0 < t < t^*$, $t^* \leq L\varepsilon^{-1}$, the inequalities

$$|x - \xi| < \eta, \qquad |\dot{x} - \dot{\xi}| < \rho \tag{8}$$

will hold.

Using continuity considerations and inequality (8), by arguing by contradiction one can show that $t^* = L\varepsilon^{-1}$.*

II. Let us now consider a system containing fast and slow variables

$$\dot{x} = \varepsilon X(t, x, \dot{x}, y, \dot{y}), \qquad \dot{y} = Y(t, x, \dot{x}, y, \dot{y}). \tag{9}$$

Here $\varepsilon > 0$ is a small parameter; x, X are n-dimensional, and y, Y are m-dimensional vectors.

Systems

$$\dot{y} = Y(t, x, \dot{x}, y, \dot{y}), \tag{10}$$

where x and $\dot{x}$ are regarded as parameters, will be called almost degenerate with respect to system (9). Suppose that the general solution

$$y = \varphi(t, x, \dot{x}, t_0, y_0), \qquad \varphi(t_0, x, \dot{x}, t_0, y_0) = y_0 \tag{11}$$

of the almost degenerate system (10) is known for arbitrary parameters $x, \dot{x}$. Let us average the function $X(t, x, \dot{x}, y, \dot{y})$ along the trajectory (11) (it is assumed that the result of the averaging does not depend on the parameters t_0, y_0):

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} X\left(t, x, \dot{x}, \varphi(t, x, \dot{x}, t_0, y_0), \frac{\partial\varphi(t, x, \dot{x}, t_0, y_0)}{\partial t}\right) = X_0(x, \dot{x}).$$

Along with system (9), we shall consider the averaged system

$$\dot{\xi} = \varepsilon X_0(\xi, \dot{\xi}), \qquad \dot{\eta} = (t, \xi, \dot{\xi}, \eta, \eta). \tag{12}$$

Let $\{x(t), y(t)\}$ be a solution of system (9), and $\{\xi(t), \eta(t)\}$ a solution of system (12), with $x(t_0) = \xi(t_0)$, $y(t_0) = \eta(t_0)$. Then, using the constructions proposed by V. M. Volosov [2], one can show that, under certain conditions, the functions $x(t)$ and $\xi(t)$ will be close on the interval $0 < t < L\varepsilon^{-1}$. In the course of the proof one uses the function $u(t, x, \dot{x}, y, \dot{y})$, determined from the equation

$$\frac{\partial u}{\partial t} = Y(t, x, \dot{x}, y, \dot{y})\frac{\partial u}{\partial y} + \left(1 - \frac{\partial Y}{\partial \dot{y}}\right)^{-1} \left(\frac{\partial Y}{\partial t} + Y\frac{\partial Y}{\partial y}\right)\frac{\partial u}{\partial \dot{y}} =$$

$$= X(t, x, \dot{x}, y, \dot{y}) - X_0(x, \dot{x}),$$

$$x = \text{const}, \qquad \dot{x} = \text{const}.$$

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References

[1] N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1963.
[2] V. M. Volosov, UMN, 17, issue 6 (1962).
[3] Yu. A. Mitropolsky, *Problems of the Asymptotic Theory of Nonstationary Oscillations*, 1964.

* For equations of the form (1), one can also indicate another averaging procedure, in which the solution of equation (1) is compared with the solution of the equation $\dot{z} = \varepsilon X_0(z, 0)$, where $X_0(z, 0)$ is the mean (with respect to t) value of the function $X(t, z, 0)$.

Note: Figure translations are in progress. See original paper for figures.