ON BEST MEAN APPROXIMATION BY POLYNOMIALS OF FUNCTIONS HAVING A REAL SINGULAR POINT

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Abstract

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ON BEST MEAN APPROXIMATION BY POLYNOMIALS OF FUNCTIONS HAVING A REAL SINGULAR POINT

(Presented by Academician S. N. Bernstein on May 6, 1966)

S. N. Bernstein ($^{1-3}$) first observed that the sequence of best uniform approximations of the function $|x-c|^p$ $(-1<c<1,\ p>0)$ on the interval $[-1,1]$ by algebraic polynomials of degree n is asymptotically equal to $(\sqrt{1-c^2})^p\mu(p)n^{-p}$ $(n\to\infty)$, where $\mu(p)$ is a certain constant, and indicated a method for finding the value of $\mu(p)$. These works play a fundamental role in the study of the properties of best approximation by polynomials of other functions with singularities of the same type.

Subsequently I. I. Ibragimov (^{4}) applied S. N. Bernstein' s method to the estimation of best uniform approximation of functions of the form $x^r|x|^\alpha\ln^m|x|$ $(r+\alpha>0,\ m\geq 0,\ r$ and m are integers, α is a real number) on the interval $[-1,1]$ by means of a polynomial of degree n; he also considered the case of the function $(a-x)^{r+\alpha}\ln^m(a-x)$ $(a\geq 1)$.

S. M. Nikol' skii (^{5}) investigated the asymptotic properties of the best approximation of the function $|a-x|^s$ by polynomials in the metric of the space L on the interval $[-1,1]$, and obtained the corresponding asymptotic equalities.

The present note is devoted to best approximation by polynomials in the metric of the space $L_q(-1,1)$ $(1\leq q\leq\infty)$ of the function $(x-a)^r\times|x-a|^\alpha\ln^m|x-a|$, where r and m are integers, and α and a are real numbers. In these investigations the main role is played by a general limit theorem of S. N. Bernstein (see (^{6}), Theorem VII bis), and also Theorem 1, generalizing the well-known method of S. N. Bernstein (^{3}) (this theorem is analogous to S. N. Bernstein' s theorems not only in formulation, but also in the method of proof).

Theorem 1 (generalization of Bernstein' s inequality). *Suppose that for the best approximation of a function $f(x)$ $(f(x)\in L_q(-2,2))$ the following conditions are satisfied*

$$1)\quad E_{n-1}(f;a,b)_{L_q}<\left(1+\frac{B\ln n}{\alpha_n}\right)E_n(f;a,b)_{L_q}\quad(-2<a<0<b<2)\quad(1)$$

where $B > 0$ is some constant; α_n is a monotonically increasing sequence $(1 \leq \alpha_n \leq n)$ such that, for any constant γ $(0 \leq \gamma < 1)$,

$$\lim_{n\to\infty} \frac{\ln n}{\alpha_n^{1-\gamma}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha_{n+o(n)}}{\alpha_n} = 1.$$

$$2) \quad E_n(x^{2p\nu} f(x); a, b)_{L_q} < \frac{C^{2\nu}(2\nu)^{2\nu}}{n^{2\nu}} E_n(f; a, b)_{L_q}, \tag{2}$$

where p and ν are natural numbers; $C > 0$ is some constant independent of ν and n.

3) There exists a natural number j such that

$$\lim_{n\to\infty} n^j E_n(f; a, b)_{L_q} = \infty. \tag{3}$$

Then

$$E_n\left(\sum_{k=1}^{l} B_k f(x - a_k); -1, 1\right)_{L_q} =$$

$$= [1 + o(1)] \left\{\sum_{k=1}^{l} |B_k|^q E_n^q(f(x - a_k); -1, 1)_{L_q}\right\}^{1/q}, \tag{4}$$

where B_k are constants, $-1 < a_k < 1,\ a_k \neq a_i\ (k \neq i),\ l < \infty$.

Consider the best weighted approximation

$$E_{n,r(x)}(f; a, b)_{L_q} = \inf_{c_k} \left\{\int_a^b \left| f(x) - \sum_{k=1}^{n} c_k x^k \right|^q [r(x)]^q \, dx\right\}^{1/q}, \tag{5}$$

where the weight $r(x)$ satisfies the following conditions:

1) $r(x) \geq \beta > 0$;

2)

$$\left\{\int_a^b [r(x)]^q \, dx\right\}^{1/q} \leq M < \infty \quad (-2 < a < 0 < b < 2);$$

3) the function $r(x)$ is continuous at the point $x = 0$.

Corollary. If the function $f(x)$ is continuous outside every neighborhood of zero and satisfies the conditions of Theorem 1, then the following asymptotic equality holds ($n \to \infty$):

$$E_{n,r(x)}(f;a,b)_{L_q} = [1+o(1)]r(0)E_n(f;a,b)_{L_q}. \tag{6}$$

Lemma. For any $0 < \delta_n < 1$, for all natural $n > 1$, on intervals $[a,b]$, where $a < 0 < b$, the inequality holds ($r+\alpha > -1/q,\ r$ and $m \geq 0$ are integers)

$$E_{n-1}(x^r|x|^\alpha \ln^m |x|;a,b)_{L_q} \leq \frac{(1+\delta_n)^{n+1/q}+1}{(1+\delta_n)^{n-r-\alpha}-1}\times$$

$$\times\left(1+\frac{C(1+\delta_n)^{n-r-\alpha}\ln(1+\delta_n)}{[(1+\delta_n)^{n+1/q}+1]\ln n}\right)E_n(x^r|x|^\alpha \ln^m |x|;a,b)_{L_q}, \tag{7}$$

where $C > 0$ is some constant.

If in this inequality we put $(1+\delta_n)^{n-r-\alpha} = 2n$, then we obtain (see ([3],[7]))

$$E_{n-1}(f;a,b)_{L_q} < \left(1+\frac{D\ln n}{n}\right)E_n(f;a,b)_{L_q}, \tag{8}$$

where $D > 0$ is some constant.

Thus, for the function $x^r|x|^\alpha \ln^m |x|$ condition (1) of Theorem 1 is satisfied. It is not difficult to verify that for this function conditions (2) with $p = 1$ and condition (3) are satisfied. Thanks to the general limit theorem of S. N. Bernstein ([6]), this makes it possible to prove the following theorem.

Theorem 2. If $r+\alpha > -1/q,\ m \geq 0$, and $|a_k| < 1$ (r and m are integers, α and a_k are real numbers), then the equality holds ($n \to \infty$)

$$E_n\left(\sum_{k=1}^{l} B_k(x-a_k)^r|x-a_k|^\alpha \ln^m |x-a_k|;-1,1\right)_{L_q} =$$

$$= [1+o(1)]\left\{\sum_{k=1}^{l}\left|B_k\left(\sqrt{1-a_k^2}\right)^{r+\alpha+1/q}\right|^q\right\}^{1/q} E_n(x^r|x|^\alpha \ln^m |x|;-1,1)_{L_q} \tag{9}$$

(B_k are constants, $l < \infty$).

If α is not an even integer, then

$$\lim_{n\to\infty}\frac{n^{r+\alpha+1/q}}{(\ln n)^m}E_n(x^r|x|^\alpha\ln^m|x|;-1,1)_{L_q}=A_1(x^r|x|^\alpha)_{L_q}<\infty; \qquad (10)$$

if α is an even integer, then

$$\lim_{n\to\infty}\frac{n^{r+\alpha+1/q}}{(\ln n)^{m-1}}E_n(x^{r+\alpha}\ln^m|x|;-1,1)_{L_q}=mA_1(x^{r+\alpha}\ln|x|)_{L_q}<\infty, \qquad (11)$$

where $A_1[f(x)]_{L_q}$ is the best approximation of the function $f(x)$ by entire functions of degree ≤ 1 in the metric of the space $L_q(-\infty,\infty)$.

Relations (9) and (10), when r is even and $m=0$, for $q=\infty$ constitute the theorem of S. N. Bernstein (^{3}), and for $q=1$ the result of S. M. Nikol' skii (^{5}). For arbitrary natural r and $m=0$, for $q=\infty$ this asymptotic equality was proved in the book of A. F. Timan (^{7}), and for arbitrary q it was given in a note (^{8}). Relations (10) and (11) were obtained by I. I. Ibragimov (^{4}) ($q=\infty$) (see also (^{9})). In the case $l=m=q=1$, $r=\alpha=0$, equalities (9) and (11) were obtained by another method by V. I. Gukevich (^{10}).

Using the corollary of Theorem 1, we obtain the theorem:

Theorem 3. Whatever the noninteger $s>-1/2q$ and the integer $m\geq 0$ may be, for the best approximation of the function $(1-x)^s\ln^m(1-x)$ in the metric of the space L_q $(1\leq q<\infty)$ with weight $(1-x^2)^{-1/2q}$ by algebraic polynomials on $[-1,1]$ there exists the limit

$$\lim_{n\to\infty}\frac{n^{2s+1/q}}{(\ln n)^m}E_{n,(1-x^2)^{-1/2q}}[(1-x)^s\ln^m(1-x);-1,1]_{L_q}=2^{m-s-1/q}A_1(|x|^{2s})_{L_q}, \qquad (12)$$

and in the case when $s\geq 0$ and $m>0$ are integers,

$$\lim_{n\to\infty}\frac{n^{2s+1/q}}{(\ln n)^{m-1}}E_{n,(1-x^2)^{-1/2q}}[(1-x)^s\ln^m(1-x);-1,1]_{L_q}=$$

$$=m2^{m-s-1/q}A_1(x^{2s}\ln|x|)_{L_q}. \qquad (13)$$

For the best approximation of the function $(a-x)^s\ln^m(a-x)$ $(a>1)$ in the metric of the space $L_q(-1,1)$, it is not difficult to obtain the inequalities

$$\frac{2^{(q+1)/q}(a^2-1)^{(s+1)/2}(\ln n)^m(1-\varepsilon_n)}{|\Gamma(-s)|\,n^{s+1}(a+\sqrt{a^2-1})^{n+2}}\leq E_n[(a-x)^s\ln^m(a-x);-1,1]_{L_q}\leq$$

$$\le \frac{2^{1/q}(a^2-1)^{(s-1)/2}(\ln n)^m(1+\varepsilon_n)}{|\Gamma(-s)|\, n^{s+1}(a+\sqrt{a^2-1})^n}, \tag{14}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, s is not a nonnegative integer, and

$$\frac{2^{(q+1)/q}(a^2-1)^{(s+1)/2} s!\, m(\ln n)^{m-1}(1-\varepsilon_n)}{n^{s+1}(a+\sqrt{a^2-1})^{n+2}} \le E_n\big[(a-x)^s \ln^m(a-x);\, -1,1\big]_{L_q} \le$$

$$\le \frac{2^{1/q}(a^2-1)^{(s-1)/2} s!\, m(\ln n)^{m-1}(1+\varepsilon_n)}{n^{s+1}(a+\sqrt{a^2-1})^n}, \tag{15}$$

if $s \ge 0$ is an integer.

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Note: Figure translations are in progress. See original paper for figures.

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