POLYHEDRALLY CLOSED SYSTEMS

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Abstract

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S. N. CHERNIKOV

POLYHEDRALLY CLOSED SYSTEMS OF LINEAR INEQUALITIES OVER AN ARBITRARY ORDERED FIELD

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1. Let $L = L(P)$ be a linear space over an arbitrary ordered field P, and let $f_\alpha(x)$ $(\alpha \in M)$ be linear (i.e., additive and homogeneous) functions defined on L, with values in P; here M is some finite or infinite set of indices.

The system

$$f_\alpha(x) - a_\alpha \leqslant 0 \qquad (\alpha \in M), \tag{1}$$

where $a_\alpha \in P$, will be called a **system of linear inequalities** on L. For system (1) with $a_\alpha = 0$, i.e., for a system of the form

$$f_\alpha(x) \leqslant 0 \qquad (\alpha \in M) \tag{2}$$

we introduce the following definition. Let L' be some subspace of the space L^* of all linear functions (with values in P) defined on L, containing the elements f_α $(\alpha \in M)$. System (2) will be called **polyhedrally** (L, L')**-closed** if its dual cone, i.e., the cone generated by the elements f_α $(\alpha \in M)$ in L', coincides with the intersection of all sets containing it that are defined in L' by inequalities of the form $x(f) \leqslant 0$ (x is a fixed element of L, and f is an element ranging over L').

Let $\bar{L}$ be the linear space of pairs $[x, t]$ $(x \in L,\ t \in P)$, $\bar{L}'$ the linear space of pairs $[f, k]$ $(f \in L',\ k \in P)$, and

$$[f, k]([x, t]) = f(x) + kt.$$

System (1) will be called **polyhedrally** (L, L')**-closed** if the system

$$f_\alpha(x) - a_\alpha t \leqslant 0 \quad (\alpha \in M), \qquad -t \leqslant 0 \tag{3}$$

is polyhedrally $(\bar{L}, \bar{L}')$-closed. A polyhedrally (L, L^*)-closed system (1) ($L' = L^*$) will be called **polyhedrally closed**. In the author' s paper ([1]) it was noted that the system

$$f_\alpha(x) - a_\alpha = a_{\alpha 1}x_1 + \dots + a_{\alpha n}x_n - a_\alpha \leqslant 0 \quad (\alpha \in M) \tag{4}$$

over the space R^n (R is the field of real numbers) is polyhedrally closed if and only if its associated cone (the cone generated by the $(n+1)$-dimensional vectors $(a_{\alpha 1}, \dots, a_{\alpha n}, -a_\alpha)$ $(\alpha \in M)$ and $(0, \dots, 0, -1)$) is topologically closed in the space R^{n+1}.

In ([1]) polyhedrally (L, L')-closed systems were considered in the case where the ground field P coincided with the field R; however, in fact, the results presented there that concern the space $L(R)$ of general form (propositions $(*)$, $(**)$, Theorems 1 and 3, and Corollary 1 of Theorem 3) are valid not only for $P = R$, but also for an arbitrary ordered field P. We note here the following generalization, following from proposition $(**)$, of the Alexandrov-Fan Ji theorem.

Theorem 1. *A polyhedrally (L, L')-closed system (1) (over $L = L(P)$) is consistent if and only if every identically*

equal to zero on the L-linear combination

$$p_{a_1} f_{a_1}(x) + \dots + p_{a_s} f_{a_s}(x)$$

(s is not fixed, $s \geqslant 1$) with positive coefficients from P, spanning a system of functions $f_{a_1}(x), \dots, f_{a_s}(x)$ of rank $s-1$, there corresponds the inequality

$$p_{a_1} a_{a_1} + \dots + p_{a_s} a_{a_s} \geqslant 0,$$

or when no such identically zero combination exists.

2. Let U be some subspace of L. The **cone** of the U-hull of the system of linear inequalities

$$f_a(x) + t_a \leqslant 0 \qquad (a \in M) \tag{5}$$

over $L = L(P)$, in which t_a are parameters taking one or another value from the field P, will be called the set $C(U)$ of nonnegative solutions $\{u_a\}$ $(a \in M)$ of the equation

$$\sum_{a \in M} u_a f_a(x) = 0 \qquad (x \in U)$$

with a finite number of nonzero coordinates u_a. An element $\{u_a^0\}$ $(a \in M)$ of the cone $C(U)$ will be called **fundamental** if the rank of the system of functions $f_a(x)$, whose indices a coincide with the indices of its nonzero coordinates u_a^0, is one less than their number. The following assertion is valid:

$(*)$

Maximal systems of essentially distinct fundamental elements of a nonzero cone $C(U)$, and only they, are its bases.

Elements of the cone $C(U)$ that differ only by positive factors (from P) are not considered essentially distinct.

If

$$C^\beta = \{C_a^\beta\} \qquad (a \in M,\ \beta \in N)$$

is some system of generating elements of the cone $C(U)$, then the system

$$\sum_{a \in M} C_a^\beta f_a(x) + \sum_{a \in M} C_a^\beta t_a \leqslant 0 \qquad (\beta \in N) \tag{6}$$

will be called the U-**hull** of system (5) (for $t_\alpha = -a_\alpha$, the U-hull of system (1)). If C^β $(\beta \in N)$ is a maximal system of essentially distinct fundamental elements of the cone $C(U)$, then we shall call the U-hull a **fundamental** U**-hull**. If the cone $C(U)$ is zero, then we shall say that the U-hull is **empty**. For $U = L$ we shall call the U-hull **complete**.

Using assertion $(*)$, it is not difficult to verify that every U-hull contains a fundamental U-hull and may differ from the latter only by inequalities that are linear combinations with positive coefficients of its inequalities.

Theorem 2. System (5) with a nonempty U-hull for some $U \subseteq L$ is compatible for those values of the parameters t_a for which it is polyhedrally (L, L')-closed and has a compatible U-hull. System (5) with an empty U-hull is compatible for arbitrary values of the parameters t_a for which it is polyhedrally (L, L')-closed.

In essence, this theorem is equivalent to the assertion that a polyhedrally (L, L')-closed system (1) with a compatible or empty U-hull is compatible.

Corollary 1. The system (4) with a conjugate cone topologically closed in the space R^{n+1} is consistent if at least one of its U-convolutions is consistent or empty for some subspace $U \subseteq L = R^n$ (for example, at least one convolution).

Theorem 3. If U is some subspace of L and V^* is the subspace of all elements $f \in L^*$ for which $f(x) = 0$ $(x \in U)$, then the dual cone of any U-convolution of the system (2) coincides with the intersection of the dual cone of the system (2) with the subspace V^*, if the U-convolution is nonempty, and with the zero element of L^*, if it is empty.

Corollary 1. If U is any subspace of L with respect to which the system (1) has a nonempty U-convolution, then from the polyhedral closedness of the system (1) there follows the polyhedral closedness of each of its U-convolutions.

Corollary 2. If the conjugate cone of the system (4) over the space R^n is topologically closed in the space R^{n+1}, then each nonempty U-convolution of it for any $U \subseteq R^n$ is polyhedrally closed.

Corollary 3. For any two subspaces U_1 and U_2 of L, the U_2-convolution of the U_1-convolution of the system (5) (of the system (1)) coincides with the $U_1 + U_2$-convolution of the latter.

3. If there exists a least upper bound T ($T \in P$) of the values of the linear function $f(x)$ on the set H of solutions of the consistent system (1), then we shall say that T is the **greatest value** of the function $f(x)$ on the set H. Obviously, T is the least upper bound of the values of the parameter t for which the system

$$f_\alpha(x) - a_\alpha \leq 0 \quad (\alpha \in M), \qquad -f(x) + t \leq 0 \tag{7}$$

is consistent.

Theorem 4. If a linear function $f(x)$ ($x \in L,\ f \in L'$) has a greatest value T on the set H of solutions of a consistent polyhedrally (L, L')-closed system (1) of nonzero rank, then the system (1) has such a finite subsystem S of rank equal to the number of its inequalities, on the set of solutions of which the greatest value of the function $f(x)$ exists, is attained, and coincides with T. If the subsystem S has no subsystems distinct from it of this kind and the value T is attained on the set H, then it is attained for those and only those solutions of the system (1) which satisfy the boundary equations of all inequalities of the subsystem S.

Theorem 5. In order that a nonzero linear function $f(x)$ ($x \in L,\ f \in L'$), having a greatest value T on the set H of solutions of a polyhedrally (L, L')-closed consistent system (1) of nonzero rank, attain it on H, it is sufficient that the corresponding system (7) with $t = T$ be polyhedrally (L, L')-closed.

The condition of the theorem is not necessary.

Example. The system

$$-\frac{1}{n(n+1)}x - y + \frac{2n+1}{n(n+1)}z \leq 0 \quad (n = 1, 2, ...),$$

$$-y \leq 0, \qquad -z \leq 0$$

is polyhedrally closed (here $L = L' = R^3$). The greatest value T of the function $-y$ on the set of its solutions is, obviously, equal to zero. It is attained for each of its solutions $(x, 0, 0)$ with $x \geq 0$. The system (7) with $t = T$ here has the form

$$-\frac{1}{n(n+1)}x - y + \frac{2n+1}{n(n+1)}z \leq 0 \quad (n = 1, 2, ...),$$

$$-y \leq 0, \qquad y \leq 0, \qquad -z \leq 0.$$

In view of Theorem 1 from the paper [1], it is not polyhedrally closed, since the inequality $z \leq 0$, which is its consequence, obviously,

cannot be represented as a linear combination with nonnegative coefficients of a finite number of its inequalities.

4. Denote by $M(P)$ the space that is the direct sum of the spaces P_α $(\alpha \in M)$, isomorphic to the space P^1. An arbitrary element x of $M(P)$ is a system $(\dots, x_\alpha, \dots)$ with a finite set of components $x_\alpha \in P_\alpha$ $(\alpha \in M)$ different from zero, containing one and only one element x_α from each P_α. If c_α $(\alpha \in M)$ is an arbitrary element of the (ordered) field P, then the expression $\sum_{\alpha \in M} c_\alpha x_\alpha$ will be called a **linear form** on the space $M(P)$. An element $x = (\dots, x_\alpha, \dots)$ of the space $M(P)$ will be called **nonnegative** if each of its components x_α is nonnegative, and **positive** if, in addition, at least one of them is positive.

The problem of minimizing the linear form $a(x) = \sum_{\alpha \in M} a_\alpha x_\alpha$ on the set N of nonnegative elements $u = (\dots, u_\alpha, \dots)$ of $M(P)$, for each of which the relation, identical with respect to $x \in L = L(P)$,

$$\sum_{\alpha \in M} u_\alpha f_\alpha(x) = f(x), \tag{8}$$

holds, will be called the **dual problem** for the problem of maximizing the linear function $f(x)$ $(x \in L,\ f \in L')$ on the set of solutions of the consistent polyhedrally (L, L')-closed system (1).

Theorem 6. *If there exists a greatest value T of the linear function $f(x)$ $(x \in L,\ f \in L')$ on the set H of solutions of a polyhedrally (L, L')-closed consistent system (1), then the set N is nonempty and on it there exists and is attained the least value of the linear form*

$$a(u) = \sum_{\alpha \in M} a_\alpha u_\alpha$$

and it coincides with T. Conversely, if the set N is nonempty and on it there exists the least value T of the linear form $a(u)$, then there exists the greatest value of the function $f(x)$ on the set H, and it coincides with T.

Thus, the linear function $f(x)$ $(x \in L,\ f \in L')$ has a greatest value on the set of solutions of the polyhedrally (L, L')-closed consistent system (1) if and only if the set N of nonnegative solutions $(\dots, u_\alpha, \dots)$ of equation (8), having a finite number of nonzero coordinates, is nonempty and on it there exists the least value of the linear form $a(u)$.

Theorem 7. *The linear function $f(x)$ $(x \in L,\ f \in L')$ has a greatest value on the set H of solutions of a polyhedrally (L, L')-closed system (1) and attains it on this set if and only if the system*

$$f_\alpha(x) - a_\alpha \leq 0 \quad (\alpha \in M), \qquad \sum_{\alpha \in M} u_\alpha f_\alpha(t) = f(t) \quad (t \in L),$$

$$\sum_{\alpha \in M} a_\alpha u_\alpha \leq f(x), \qquad -u_\alpha \leq 0$$

has at least one solution (x, u), where $x \in H$, and u is a nonnegative vector $(\dots, u_\alpha, \dots)$ with a finite number of positive coordinates.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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REFERENCES

1. S. N. Chernikov, *DAN*, **161**, No. 1, 55 (1965).

Note: Figure translations are in progress. See original paper for figures.