

# GROUPS WITH COMPLEMENTED NONINVARIANT SUBGROUPS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **GROUPS WITH COMPLEMENTED NONINVARIANT SUBGROUPS**

*(Presented by Academician A. I. Mal' tsev on 10 III 1966)*

Numerous works have been devoted to the study of groups that are factorable in some sense, and these investigations occupy an important place in group theory. But the theory of factorable groups is so extensive that it can hardly be regarded as complete. As early as 1937, P. Hall, in a classical note <sup>(1)</sup>, studied finite groups all of whose subgroups are complemented. Such groups, thanks to the work of S. N. Chernikov and his students, entered the literature under the name of completely factorable groups. In this connection a subgroup  $H$  is called a **complemented** subgroup in a group  $G$  if there exists in  $G$  a subgroup  $F$  such that  $G = H \cdot F$  and  $H \cap F = \{1\}$ .

S. N. Chernikov, in papers <sup>(2,3)</sup>, proposed an extensive program for the study of groups with prescribed systems of complemented subgroups. A group  $G$  possessing one or another system of subgroups each of which is complemented in  $G$  is called **factorable with respect to this system of subgroups**. M. I. Kargapolov, in works <sup>(4,5)</sup>, solved a number of fundamental group-theoretic and, in particular, factorization problems. In our view, the ideas of <sup>(2-5)</sup> are far from exhausted. N. V. Chernikova, in <sup>(6,7)</sup>, gave a description of infinite completely factorable groups. Later Yu. M. Gorchakov, in <sup>(8,9)</sup>, generalized the results of <sup>(1,6,7)</sup>, studying primitively factorable groups.

Yu. M. Gorchakov suggested to the author that he investigate groups factorable with respect to the system of all noninvariant subgroups. It is clear that the class of such groups includes the classes of abelian, Hamiltonian, and completely factorable groups. Since completely factorable groups have been completely studied in <sup>(1,6,7)</sup>, they are considered known, and the description of groups factorable with respect to the system of all noninvariant subgroups is given up to completely factorable groups. Finite groups all of whose noninvariant subgroups are complemented were studied in <sup>(10)</sup>.

The present work is devoted to the formulation of the following four theorems, which describe and classify exhaustively the groups factorable with respect to the system of all noninvariant subgroups.

**Theorem 1.** *A group  $G$ , factorable with respect to the system of all noninvariant subgroups, is completely factorable if and only if all Sylow  $p$ -subgroups of  $G$  are elementary abelian.*

This theorem is analogous to Theorem 14 of <sup>(3)</sup> and Theorem 4 of <sup>(4)</sup>. For finite groups Theorem 1 follows immediately from the result of V. Gaschütz <sup>(11)</sup> on the factorability of a finite group with elementary abelian Sylow  $p$ -subgroups with respect to the system of all invariant subgroups.

**Theorem 2.** *A  $p$ -group is factorable with respect to the system of all noninvariant subgroups if and only if it is either abelian, or has the form*

$$G = P \times A,$$

where  $A$  is an elementary abelian group, and  $P$  is a group of one of the types:

1°. A  $p$ -group with commutator subgroup of order  $p$  and with center possessing a unique subgroup of order  $p$ .

2°.  $P = Q \times \{c\}$ , where  $c^4 = 1$ , and  $Q$  is the quaternion group of order 8.

3°.  $P = \{a\} \lambda \{b\}$ ;  $a^{p^m} = b^{p^n} = 1$ ;  $m, n$  are natural numbers,  $m \geq 2$ ,  $n \geq 2$ ,  $m + 1 \geq n$ ,  $[a, b] = a^{p^{m-1}}$ .

4°.  $P = (\{a_1\} \times \{a_2\} \times \{z_1\} \times \{z_2\}) \lambda \{a\}$ ,  $a_i^p = z_i^p = a^p = 1$ ,  $[a_i, a] = z_i$ ,  $[a, z_i] = 1$  ( $i = 1, 2$ ).

5°.  $P = ((\{a_1\} \times \{z_1\} \times \{z_2\} \times \{z_3\}) \lambda \{a_2\}) \lambda \{a_3\}$ ,  $a_i^p = z_i^p = 1$ ,  $[a_1, a_2] = z_1$ ,  $[a_1, a_3] = z_2$ ,  $[a_2, a_3] = z_3$ ,  $[a_i, z_j] = 1$  ( $i, j = 1, 2, 3$ ).

6°.  $P = (\{a\} \times \{z\}) \lambda \{b\}$ ,  $a^4 = b^4 = z^2 = 1$ ,  $[a, b] = z$ ,  $[b, z] = 1$ .

7°.  $P = (\{a\} \times \{z\}) \lambda \{b\}$ ,  $a^4 = b^2 = z^2 = 1$ ,  $[a, b] = z$ ,  $[b, z] = 1$ .

8°.  $P = (\{a\} \times \{b\}) \lambda \{c\}$ ,  $a^4 = b^4 = c^2 = 1$ ,  $[a, c] = a^2$ ,  $[b, c] = b^2$ .

**Theorem 3.** A nilpotent group is factorizable with respect to the system of all noninvariant subgroups if and only if it is either abelian, or Hamiltonian, or has the form

$$G = P \times A,$$

where  $A$  is a completely factorizable abelian group\*, and  $P$  is a group of one of the types 1°-8° of Theorem 2.

**Theorem 4.** A nonnilpotent and not completely factorizable group is factorizable with respect to the system of all noninvariant subgroups if and only if it has the form

$$G = A \lambda P,$$

where:

1°.  $A$  is a completely factorizable abelian group and every subgroup of  $A$  is invariant in  $G$ .

2°.  $P$  is a Sylow  $p$ -subgroup of  $G$ , and for every subgroup  $\{t\} \subseteq P$  not contained in the centralizer of  $A$ , there exists an elementary abelian complement in  $P$ .

If  $A_q$  is an arbitrary Sylow  $q$ -subgroup of  $A$ , then transformation of  $A_q$  by an element  $t$  is equivalent to raising all elements of  $A_q$  to one and the same definite power.

3°.  $P = Q \lambda R$ , where  $R$  is an elementary abelian group, and  $Q$  is a group of one of the types:

- 1)  $Q = (\{a\} \times \{z\}) \lambda \{b\}$ ,  $a^p = b^p = z^p = 1$ ,  $[a, b] = z$ ,  $[b, z] = 1$ .
- 2)  $Q = \{a\} \lambda \{b\}$ ,  $a^{p^m} = b^p = 1$ ,  $[a, b] = a^{p^{m-1}}$ ;  $m \geq 2$  is a natural number.
- 3)  $Q = (\{a_1\} \times \{a_2\} \times \{z_1\} \times \{z_2\}) \lambda \{a\}$ ,  $a_i^p = z_i^p = a^p = 1$ ,  $[a_i, a] = z_i$ ,  $[a, z_i] = 1$  ( $i = 1, 2$ ).
- 4)  $Q = (\{a\} \times \{z\}) \lambda \{b\}$ ,  $a^4 = b^2 = z^2 = 1$ ,  $[a, b] = z$ ,  $[b, z] = 1$ .
- 5)  $Q = \{a\}$ ,  $a$  is an element of nonprime order.  $\lambda$  is the sign of a semidirect product,  $p, q$  are prime numbers.

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- \* A completely factorizable abelian group is a direct product of elementary abelian groups.

*Note: Figure translations are in progress. See original paper for figures.*

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