SYMMETRIC ALGEBRAS AND STEINER SYSTEMS

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SYMMETRIC ALGEBRAS AND STEINER SYSTEMS

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In this paper certain combinatorial designs are considered, i.e., combinations of elements of a given set possessing specified properties, and an algebraic interpretation of these designs is given. Such designs are Steiner triples and quadruples –they correspond to TS-quasigroups ([1]) and S-algebras ([2]). On the set of finite S-algebras a binary operation is defined, distinct from the direct product.

Let a Steiner triple system M be given on a set A, i.e., a system of triples (x, y, z) of pairwise distinct elements of A, such that any pair $x, y \in A$ $(x \neq y)$ occurs in one and only one triple of M. Adjoin to the system M the trivial triples (x, x, x); then we obtain the system $\overline{M}$. Define multiplication on A by putting $xy = z \Longleftrightarrow (x, y, z) \in \overline{M}$. Then A becomes a TS-quasigroup ([1]), i.e., a quasigroup with the identities

$$x^2 = x, \qquad (xy)y = x, \qquad x(xy) = y.$$

Obviously the converse is also true: if A is a TS-quasigroup, then on the set A one can define a Steiner triple system by putting $(x, y, z) \in M \Longleftrightarrow xy = z$.

Adjoin to the TS-quasigroup A an element e and define on the set $\bar{A} = A \cup \{e\}$ a new operation $(\cdot)$, putting, for arbitrary $x, y \in A$ $(x \neq y)$, $x \cdot y = xy$, $x \cdot e = e \cdot x = x$, $x \cdot x = e$, $e \cdot e = e$. Then $\bar{A}$ becomes a TS-loop with identity e, i.e., a loop with the identities $x^2 = e$, $(xy)y = x$, $x(xy) = y$. The correspondence obtained between TS-quasigroups and TS-loops is, obviously, invertible; thus the following assertion is true.

Proposition 1. *There exists a one-to-one correspondence between the sets: I–Steiner triple systems of order n; II–TS-quasigroups of order n; III–TS-loops of order $n+1$.*

Let a Steiner quadruple system N be given on a set B, i.e., a system of quadruples (x, y, z, t) of pairwise distinct elements of B, such that any triple $x, y, z \in A$ $(x \neq y \neq z \neq x)$ occurs in one and only one quadruple of N. Adjoin to the system all possible quadruples of the form (x, x, y, y); then we obtain the system $\overline{N}$.

Define on B a ternary operation $xyzt$, putting

$$xyzt = t \Longleftrightarrow (x, y, z, t) \in \overline{N}.$$

With respect to this operation B is an S-algebra in the sense of ([2]), i.e., an algebra with one basic ternary operation $xyzt$, satisfying the identities

$$xxyt = y, \qquad xyzyzt = x, \qquad xxyztzt = y, \qquad xyxyzt\tau = z. \tag{1}$$

We note that the identities (1) may also be written in the form of conditional identities

$$x = y \Rightarrow xyzt = z, \qquad xyzt = t \Rightarrow tyzt = x \;\&\; xtzt = y \;\&\; xytt = z$$

and imply the commutativity of the basic operation

$$xyzt = xzyt = zxy\tau.$$

It is easy to see that to each S-algebra B there corresponds a certain Steiner system of quadruples on the set B. Thus, the following assertion is true.

Proposition 2. *There exists a one-to-one correspondence between the set of Steiner systems of quadruples of order n and the set of S-algebras of order n.*

The class of S-algebras is determined by the identities (1); consequently, this class is a variety and, in particular, is closed under direct (Cartesian) products. On this class we shall define another operation, which we shall call the **product of S-algebras with an identified element** (or the matrix product). The construction of this product will be set out for the case of finite S-algebras, but it is applicable without any changes also to infinite S-algebras.

Let A be an S-algebra of order t_1+1, and B an S-algebra of order t_2+1. Choose one element in each of A and B and identify them; denote this common element by ε. Denote the remaining elements of A by a_i, and the elements of B by b_i $(1 \le i \le t_1,\ 1 \le j \le t_2)$. Let

$$C = \{\varepsilon\} \cup \{c_{ij},\ 1 \le i \le t_1,\ 1 \le j \le t_2\}, \quad \bar{C} = \|c_{ij}\|.$$

On the set C we shall define a Steiner system of quadruples P, with which the construction of an S-algebra on C is associated; the resulting S-algebra of order $t_1 t_2 + 1$ will be the desired one. The construction of the system P is divided into three stages.

I. $t_1 = t_2 = 3$.

a) The quadruples from P containing ε will be the quadruples of the form

$$(\varepsilon,\ c_{i1},\ c_{i2},\ c_{i3}),\quad (\varepsilon,\ c_{1j},\ c_{2j},\ c_{3j})\quad (i,j=1,2,3),$$

as well as quadruples of the form

$$(\varepsilon,\ c_{i_1j_1},\ c_{i_2j_2},\ c_{i_3j_3}),$$

where (i_1,i_2,i_3), (j_1,j_2,j_3) are permutations of the numbers $1,2,3$; the number of such quadruples is 12.

b) Combine into one quadruple the elements of any minor of order 2 of the matrix $\bar{C}$; then we obtain another 9 quadruples.

c) To each element c_{ij} assign the quadruple of elements located in the i-th row and in the j-th column of the matrix $\bar{C}$ and distinct from c_{ij}; this rule gives another 9 quadruples.

Thus, 30 quadruples are obtained, and they form the complete system P; we shall denote this system below by P_3.

Remark. Every Steiner system of quadruples of order 10 is isomorphic to P_3.

II. $t_1 > 3,\ t_2 = t > 3$.

a) Let (j_1,j_2,j_3) be an arbitrary triple of numbers for which

$$b_{j_1}b_{j_2}b_{j_3}\tau=\varepsilon.$$

Then consider the minor of order (3×3) situated in the columns of the matrix $\bar{C}$ with numbers j_1,j_2,j_3. Associate with this minor (and with the element ε) the system P_3; then we obtain a certain set of quadruples, in particular all quadruples from P containing ε.

b) Let (j_1,j_2,j_3,j_4) be a quadruple of numbers for which

$$b_{j_1}b_{j_2}b_{j_3}\tau=b_{j_4}.$$

Then consider the minor of order (3×4) situated in the columns of the matrix $\bar{C}$ with numbers j_1,j_2,j_3,j_4. Since the numbers j_k $(k=1,2,3,4)$ are distinct, we may, without loss of generality, regard them as ordered increasingly:

$$j_1<j_2<j_3<j_4.$$

From the elements of the indicated minor we construct quadruples of the form

$$(c_{i_1j_1},\ c_{i_2j_2},\ c_{i_3j_3},\ c_{i_4j_4}),$$

where

$$(i_1,i_2,i_3,i_4)=(i,j,j,i),\ (i,j,i,j)\quad (i,j=1,2,3);$$
$$(i,i,j,k),\ (i,j,k,k)=(j_1,j_2,j_3,j_4),$$

where (i,j,k) is a permutation of the numbers $1,2,3$.

Rules a), b) give the complete Steiner system of quadruples over C; denote it by P_t. Analogously, the Steiner quadruples are constructed in the case $t_1 = t > 3,\ t_2 = 3$; we shall denote this system by P'_t $(P_3 = P'_3)$.

Remark. The construction of the system $P_t(P'_t)$ depends on the structure of the algebra $B(A)$, and therefore one ought to write $P_t(B)$, $P'_t(A)$. We use the abbreviated notation, since from the context it is clear which algebras are meant.

III. $t_1, t_3 > 3$.

a) Let (i_1, i_2, i_3) be an arbitrary triple of numbers for which

$$a_{i_1} a_{i_2} a_{i_3} \tau = \varepsilon.$$

Consider the minor of order $3 \times t_2$, situated in the rows

the matrix $\bar{C}$ with numbers i_1, i_2, i_3. Associate with this minor (and with the element ε) the Steiner system P_{t_2}. Similarly, if (j_1, j_2, j_3) is a triple of numbers for which $b_{j_1} b_{j_2} b_{j_3} t = \varepsilon$, associate with the minor of the matrix $\bar{C}$ situated in the columns with numbers j_1, j_2, j_3 the system of quadruples P'_{t_1}.

b) Let (i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4) be quadruples of pairwise distinct numbers for which

$$a_{i_1} a_{i_2} a_{i_3} \tau = a_{i_4}, \qquad b_{j_1} b_{j_2} b_{j_3} \tau = b_{j_4}.$$

Then put

$$(c_{i_1 j_1}, c_{i_2 j_2}, c_{i_3 j_3}, c_{i_4 j_4}) \in P.$$

Rules a), b) exhaust the construction of the Steiner system of quadruples. In each of the cases I–III, denote by $A \circ B$ the S-algebra corresponding to the constructed system of quadruples. We summarize the results of the construction described above in the following theorem.

Theorem. Let A_i $(i = 1, 2)$ be an S-algebra of order $t_i + 1$, $t_i \geq 3$. Then there exists an S-algebra of order $t_1 t_2 + 1$, containing t_1 subalgebras isomorphic to A_2, t_2 subalgebras isomorphic to A_1, $t_1(t_1 - 1)/6$ subalgebras isomorphic to P_{t_2}, $t_2(t_2 - 1)/6$ subalgebras isomorphic to P'_{t_1}, and

$$t_1 t_2 (t_1 - 1)(t_2 - 1)/36$$

subalgebras of order 10.

To prove the theorem it is enough to consider the product $A_1 \circ A_2$ of the S-algebras A_1, A_2 with identified element. Note that the S-algebras $A_1 \circ A_2$ and $A_2 \circ A_1$ are isomorphic. However, the question of whether the product $A_1 \circ A_2$ depends on the choice of the elements in A_1, A_2 subject to identification remains open. In this connection, the question of whether the TS-loops obtained from a given S-algebra A by introducing the derived operation $x \cdot y = xy\alpha$, as the element varies, will be isomorphic to one another is also of interest.

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Note: Figure translations are in progress. See original paper for figures.