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Abstract

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ON ONE TYPE OF CONTRACTIBLE CURVES

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Let the curve $A = \cup A_i$, where the A_i are irreducible components, lie on a nonsingular complex surface X. The curve A is called contractible (exceptional) if there exists a holomorphic mapping π of the surface X into a complex space Y, which takes the whole curve A to one point $p \in Y$ and is biholomorphic on $X \setminus A$. In Grauert' s paper (^{1}) it was proved that the curve A is contractible if and only if the intersection matrix (A_i, A_j) is negative definite. In the present note we consider contractible curves preserving the canonical class of the surface, i.e. such that the canonical bundle of the surface X is trivial in some neighborhood of the curve A. M. Artin proved in (^{2}) that for this it is necessary and sufficient that each component A_i of the curve A be a nonsingular rational curve with $A_i^2 = -2$, and that in this case the curve A is contractible also in the algebraic sense. As was shown in another paper of M. Artin (^{3}), the result of contracting such curves is rational double singularities, and only these.

In the present note it is shown that rational double points are nonvarying, and if neighborhoods of two rational double points are homeomorphic, then they are also biholomorphically equivalent. It follows that any rational double point can be realized as a singular point on a surface V in three-dimensional space, defined by an equation $f(x, y, z) = 0$ of a certain form. For the resolution of this singularity it proves sufficient to apply a sequence of σ-processes with centers at singular points of multiplicity 2. Let $\widetilde{V}$ be the nonsingular surface obtained after such a resolution. From the last theorem of the present note it follows that the surface $\widetilde{V}$ is diffeomorphic to the nonsingular surface $f(x, y, z) = c$.

1. Let the curve $A = \cup A_i$, corresponding to a rational double point, lie on a nonsingular surface X, and let the curve $\widetilde{A} = \cup \widetilde{A}_i$ isomorphic to it lie on a nonsingular surface $\widetilde{X}$.

Theorem 1. *Neighborhoods of the curve A on the surface X and of the curve $\widetilde{A}$ on the surface $\widetilde{X}$ are biholomorphically equivalent.*

Proof. Let $\mathfrak{m}_i$ be the sheaf of ideals on the surface X corresponding to the curve A_i, and $\mathcal{O}(X)$ the sheaf of germs of holomorphic functions on X.

Denote by $(A, \mathcal{O}_Z)$ the complex space with structure sheaf

$$\mathcal{O}_Z = \mathcal{O}_X/\mathfrak{m}(Z) \mid A,$$

where $Z = \sum k_i A_i,\ k_i > 0$, and

$$\mathfrak{m}(Z) = \mathfrak{m}_1^{k_1}\mathfrak{m}_2^{k_2} \dots \mathfrak{m}_n^{k_n}.$$

As follows from Theorem 3 of (^{4}), in order to prove Theorem 1 it is enough to show that the complex spaces $(A, \mathcal{O}_{NA})$ and $(\widetilde{A}, \mathcal{O}_{N\widetilde{A}})$ are isomorphic for sufficiently large N. The isomorphism of the curves A and $\widetilde{A}$ is an isomorphism of the complex spaces $(A, \mathcal{O}_A)$ and $(\widetilde{A}, \mathcal{O}_{\widetilde{A}})$.

Suppose that some isomorphism

$$\varphi:\ (A, \mathcal{O}_Z) \to (\widetilde{A}, \mathcal{O}_{\widetilde{Z}})$$

has been established. How can one establish an isomorphism of the complex spaces $(A, \mathcal{O}_{Z+A_i})$ and $(\widetilde{A}, \mathcal{O}_{\widetilde{Z}+\widetilde{A}_i})$? The obstruction $\gamma(\varphi)$ to extending the given isomorphism φ, as computed in (^{1}), lies in the group

$$H^1(A_i, \Theta \otimes (-Z, A_i)),$$

where Θ is the restric-

of the tangent bundle of the surface X on the curve A_i, and $-Z \cdot A_i$ is the bundle corresponding to the cycle $-Z$, restricted to A_i. If this group is trivial, then there exists an isomorphism of complex spaces $(A, \mathcal{O}_{Z+A_i})$ and $(\widetilde{A}, \mathcal{O}_{\widetilde{Z}+\widetilde{A}_i})$ extending the given isomorphism φ. Suppose that this group is nontrivial and $\gamma(\varphi) \neq 0$. Consider the mapping

$$\delta_Z : \mathrm{Aut}(A, \mathcal{O}_Z) \to H^1(A_i, \Theta \otimes (-Z, A_i)),$$

which assigns to each automorphism of the complex space $(A, \mathcal{O}_Z)$ the obstruction to extending it to an automorphism of the space $(A, \mathcal{O}_{Z+A})$. If $\delta_Z\alpha = \gamma(\varphi)$, then the isomorphism $\varphi\alpha^{-1}$ can be extended. If the image of the mapping δ_Z coincides with the group $H^1(A_i, \Theta \otimes (-Z, A_i))$, then from the isomorphism of the complex spaces $(A, \mathcal{O}_Z)$ and $(\widetilde{A}, \mathcal{O}_{\widetilde{Z}})$ there follows an isomorphism of $(\widetilde{A}, \mathcal{O}_{Z+A_i})$ and $(\widetilde{A}, \mathcal{O}_{\widetilde{Z}+\widetilde{A}_i})$.

In the following two lemmas we consider contractible curves corresponding to rational singular points (for the definition see (2)). By K is denoted the canonical divisor on the surface X. In the case of a double rational point, $K \cdot A_i = 0$ for every i. It is also assumed that the intersection matrices of the curves A and $\widetilde{A}$ coincide.

Lemma 1. If $(Z + K)A_i < A_i(A - A_i)$ and there exist curves A_k, where $k = 1, 2, \dots, (Z + K)A_i + 1$, such that $A_k \cdot A_i = 1$ and $(Z + K) \cdot A_k \leq 0$, then the image of the mapping δ_Z coincides with the group $H^1(A_i, \Theta \otimes (-Z, A_i))$.

Lemma 2. If $(Z+K)A_i \leq 1$ and there exist curves $A_1, \dots, A_k$ such that $A_i \cdot A_1 = A_1 \cdot A_2 = \dots = A_{k-1} \cdot A_k = 1$ and $((Z+K)-B) \cdot A_l \leq 0$, where $B = A_1 + A_2 + \dots + A_k$, for all $l = 1, \dots, k;\ i$, then the image δ_Z coincides with the whole group $H^1(A_i, \Theta \otimes (-Z, A_i))$.

Remark 1. If $k = 0$, then the lemma asserts the surjectivity of δ_Z in the case $(Z+K)A_i = 0$.

Remark 2. In considering analogous questions for algebraic varieties over a field of characteristic $p \neq 0$, Lemma 1 remains valid for any p, while Lemma 2 is valid only for p relatively prime to the determinant $|A_l \cdot A_j|$, where $l, j = 1, \dots, k;\ i$.

For curves corresponding to double rational points, one can construct a chain of cycles $Z_{m+1} = Z_m + A_{i(m)}$, satisfying, for every m, either Lemma 1 or Lemma 2, beginning with the cycle $Z_0 = A$, and such that for every N there is a cycle Z_m in the chain satisfying the inequality $Z_m > NA$. Hence the assertion of Theorem 1 follows. By computing the fundamental group it is not hard to show that the boundaries of neighborhoods of different rational double points are not homeomorphic.

2. In (2) all possible configurations of the curves under consideration are listed. It can be shown that, resolving the singularity given in C^3 by the equation $x^2+y^2+z^n = 0$, we obtain a curve of type A_{n-1}; the equation $x^2+y^2z+z^n = 0$ corresponds to the curve B_{n+1}; the equations $x^2+y^3+z^4 = 0$, $x^2+y^3+yz^3 = 0$, $x^2+y^3+z^5 = 0$ correspond to the curves E_6, E_7, and E_8. From Theorem 1 it then follows that a sufficiently small neighborhood of any rational double point can be biholomorphically embedded in C^3 and is defined there by one of the equations given above. It is also easy to observe that, for resolving these singularities, it is enough to apply σ-processes with centers at singular points of multiplicity 2. It is not hard to show the converse as well: if, in resolving some singularity on a surface in C^3, only σ-processes with centers at singular points of multiplicity 2 are applied, then it is a rational double point.

3. Consider in the space C^3 the surface F_c defined by the equation $f(x,y,z) = c$. Suppose that for $c \neq 0$ the surface F_c is nonsingular, while the surface F_0 has a unique singular point at the origin. Let $\widetilde{F}_0$ be the nonsingular surface obtained as the result of resolving the singularity on F_0.

Theorem 2. If, in resolving the singularity on the surface F_0, only σ-processes with centers at singular points of multiplicity 2 are applied, then the surface $\widetilde{F}_0$ is diffeomorphic to the surface F_c.

Proof. In the simplest case, when $f(x,y,z) = x^2+y^2+z^2$, the required diffeomorphism is easily constructed explicitly. In the general case we carry out induction on the number of σ-processes needed to resolve the singularity. Suppose, for definiteness, that the quadratic part of the function $f(x,y,z)$ has rank 1; then we may assume that

$$f(x, y, z) = x^2 + g(x, y, z),$$

where the function $g(x, y, z)$ has a zero of multiplicity 3 at the origin. Consider the family of surfaces $F(c_1, c_2, c)$ given by the equations

$$f(x, y, z) = c_1 y^2 + c_2 z^2 + c.$$

The nonsingular surfaces $F(c_1, c_2, c)$ are diffeomorphic to one another and, in particular, to the surfaces $F_c = F(0, 0, c)$. For $c_1 = c_2 = c = 0$ we obtain the surface F_0; for $c = 0,\ c_1 \neq 0,\ c_2 \neq 0$ we obtain a surface with one quadratic singularity. Apply a σ-process with center at the origin. The surface σF_0 has singularities whose resolution requires applying a smaller number of σ-processes. Using the induction hypothesis, one can show that the singular surface $\tilde{F}_0$ is diffeomorphic to the nonsingular surface $\sigma F(c_1, c_2, 0) \overset{\text{diff}}{\approx} F(c_1, c_2, c)$ for c_1 and $c_2 = 0$; hence the assertion of the theorem follows.

4. Let $\mathcal{V}$ be a singular n-dimensional complex variety, decomposing by means of a holomorphic mapping $\pi : \mathcal{V} \to T$ into complex surfaces V_t. Suppose that the base T is nonsingular, and all fibers V_t are either nonsingular or have only double rational singularities, while the set $S \subset T$ of those points t for which the surface V_t is singular is a complex subspace of smaller dimension. Let $s \in S$. Resolve the singularities on V_s minimally. We obtain the surface $\tilde{V}_s$. From Theorems 1 and 2 one can derive the following assertion:

Theorem 3. The surface $\tilde{V}_s$ is diffeomorphic to the nonsingular surface V_t.

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