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Abstract

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ON ONE MODEL OF A CONTINUOUS MEDIUM

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The state of a continuous medium—a liquid or a gas—is characterized by the distribution densities of various physical quantities: mass, energy, and momentum components. These quantities, as functions of time t and space x, must satisfy the well-known equations of hydrodynamics. If dissipative effects (viscosity, etc.) are not taken into account, then each of these equations can be written in the form

$$\partial\rho/\partial t + \operatorname{div} f = 0, \tag{1}$$

where ρ is the distribution density of one of the above-mentioned quantities (mass, energy, a momentum component), and f is the vector of flux density of this quantity. Here f is a known function of the distribution densities.

It is customary to regard a continuous medium as a set of particles, each of infinitesimally small mass, filling some volume. These particles move in space, carrying mass from one point to another. The equation of motion of a particle in the notation already used is naturally written in the form

$$\rho\, dx/dt = f, \tag{2}$$

where ρ is the distribution density of mass and f is the flux density of mass. But a continuous medium admits such an interpretation not only with respect to mass, but also to energy and momentum. In the mathematical description of the process—in equations (1)—mass, energy, and the momentum components occupy an identical position. Therefore one may imagine a continuous medium as a set of particles (or "quanta") of various kinds—particles of mass, particles of energy, particles of momentum components. A change in the state of the medium reduces to the displacement of all these particles in space. The trajectory of a particle of each kind is determined by the equation of motion (2), where ρ and f are the distribution density and flux density of that physical quantity which the given particle represents. Of course, by virtue of the definition of ρ as a

Fig. 1

Figure 1: Fig. 1

distribution density, a system of equations of the form (2) is not a system of ordinary differential equations. It is equivalent to a system of equations of the form (1), and solving it is no easier.

However, the model described may prove useful, for example, in passing to a discrete approximation of a continuous medium and in constructing numerical methods for solving problems connected with the integration of systems of the form (1). In this case the number of particles of each kind is finite, and the mass (respectively, energy or momentum) of each particle is also finite. It is necessary to indicate a method for computing the distribution density ρ from a discrete set of particles. Let k be the number of a particle, x_k its position, m_k its mass (energy, momentum component), and h a parameter of the order of the distance between particles. Then $\rho(x)$ can be defined as follows:

$$\rho(x) = \sum_k \delta_h(x_k - x)\, m_k, \tag{3}$$

where $\delta_h(x)$ is a delta-like function tending to $\delta(x)$ as $h \to 0$, for example:

$$\delta_h(x) = \begin{cases} c_h(1 - |x|/h), & \text{for } |x| < h, \\ 0, & \text{for } |x| > h; \end{cases} \tag{4}$$

here c_h is a normalizing constant. It is easy to see that in the limit (a continuous medium) formula (3) becomes the usual definition of the distribution density

$$\rho(x) = \int \delta(x' - x)\, dM(V') = \frac{dM}{dV}, \tag{5}$$

where $M(V)$ is the mass (energy, momentum component) of the volume V. If the initial positions of particles of all kinds are specified, then, solving the system —now of ordinary differential equations of the form (2)—we find the positions of the particles and, together with them, the distribution density at subsequent times.

The computations that have been carried out for certain problems—unfortunately, so far only one-dimensional ones—have shown the fundamental possibility of using the proposed model for constructing numerical methods.

Fig. 1

As an illustration, we shall describe the computation of the following problem. One-dimensional isentropic motion of a gas is described by the system of equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left(p + \rho u^2\right) = 0, \tag{6}$$

where ρ is the density, u the velocity, and p the pressure; in this computation $p = 14\rho^3$. The initial data are

$$\rho = 1, \quad u = -6 \quad \text{for } x < 0,$$

$$\rho = 2, \quad u = 1 \quad \text{for } x > 0. \tag{7}$$

In order not to deal with momentum that changes sign, as the distribution density of momentum we take not ρu, but $\rho u + 14 = \sigma$. Then equations (6) are rewritten in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \sigma}{\partial x} = 0,$$

$$\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x}\left(14\rho^3 + \frac{1}{\rho}(\sigma - 14)^2\right) = 0, \tag{8}$$

and the initial data (7) in the form

$$\rho = 1, \quad \sigma = 8 \quad \text{for } x < 0,$$

$$\rho = 2, \quad \sigma = 16 \quad \text{for } x > 0. \tag{9}$$

To compute the distribution densities ρ and σ, we shall use formulas (3), (4) with $c_h = 1/h$. We set $m_k = 1$ for the mass particles,

$m_k = 8$ for the momentum particles, $h = 16$ in both cases. At the initial moment we place the particles with spacing 1 for $x < 0$ and with spacing 0.5 for $x > 0$. This realizes the initial data (9). The equations of motion (2) are written as follows (the first for mass, the second for momentum):

$$\rho\, dx/dt = \sigma - 8,$$

$$\sigma\, dx/dt = 14\rho^3 + \frac{1}{\rho}(\sigma - 14)^2 - 50. \tag{10}$$

The right-hand sides in (10) have been reduced by constants (8 and 50) for convenience in organizing the computations. This changes the trajectories but, of course, does not affect the functions sought, $\rho(x,t)$, $\sigma(x,t)$.

The system of equations (10) was integrated by the simplest method (Euler polygons) with time step $\Delta t = 1/16$. Figure 1 shows the distribution of ρ obtained as a result of the computation at the 160th step, $t = 10$. For comparison, the exact solution is given (dashed line).

Tests of the method carried out on flows with a shock wave, as well as with another form of the function $\delta_h(x)$, gave analogous results.

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Note: Figure translations are in progress. See original paper for figures.