ON VERBAL SUBGROUPS OF FREE GROUPS

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Abstract

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ON VERBAL SUBGROUPS OF FREE GROUPS

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In the present paper the following question is answered affirmatively: does the inclusion $V(R) \supseteq V(S)$ imply the inclusion $R \supseteq S$, if $\mathfrak{B}$ is some variety of groups distinct from $\mathfrak{E}$, the variety of all groups, and R and S are normal divisors of an arbitrary non-abelian free group F. This assertion is a generalization of the well-known theorem of Auslander and Lyndon ([1]) for the particular case of the variety of all abelian groups. Other special cases were considered by various authors ([2–5]). Related questions were considered by T. Taylor and M. Danbud ([6]). The history and bibliography of the question are presented most fully in ([5]). In its ideas and methods the present work is close to ([5]).

Let $\mathfrak{B}$ be some variety of groups; let a group G be $\mathfrak{B}$-free and let the elements $y_1, y_2, \dots, y_m$ form its $\mathfrak{B}$-basis. We shall say that the elements $y_1, y_2, \dots, y_m$ multiply $\mathfrak{B}$-freely and that the group G is $\mathfrak{B}$-freely generated by the elements $y_1, y_2, \dots, y_m$.

Theorem. *Let F be a non-abelian free group; let R and S be its normal divisors such that $S \supset R$, and let $\mathfrak{B}$ be a variety of groups such that $V(S) \subseteq V(R)$; then $\mathfrak{B}$ coincides with the variety $\mathfrak{E}$ of all groups.*

Obviously, this theorem is equivalent to the result formulated above.

In ([5]) P. M. Neumann showed that it is enough to prove the theorem in the following special case: F is the free group of rank 2 with free basis $\langle a, b\rangle$; $R = \{a^p, b\}^F$, where p is a prime number; the normal divisor S is such that $F = SR$. Hence it follows that in R there is an element z such that $S \ni c = az$. Clearly, $c = a^{pn+1} b^m w$, where $w \in F'$.

Next, following the ideas of P. M. Neumann ([5]), we construct a variety $\mathfrak{D}$ having the following properties: $D(R) = D(S)$; $\mathfrak{D} \subseteq \mathfrak{B}$; $\mathfrak{D}$ is generated by finite p-groups; hence, in particular, it follows that all free groups of this variety are approximated by finite p-groups. Obviously, it suffices for us to prove $\mathfrak{D} = \mathfrak{E}$.

Lemma 1. *Let $H = F_k(\mathfrak{D})$ be the free group of the variety $\mathfrak{D}$ of finite rank k, and let $y_1, y_2, \dots, y_m$ be elements of the group H, independent modulo the commutant H' and such that the factor group $H/H'\{y_1, \dots, y_m\}$ is a finite group of order relatively prime to p. Then the elements $y_1, y_2, \dots, y_m$ multiply $\mathfrak{D}$-freely.*

For the proof we refer to Lemma 2.4 of ([5]).

Put $A = F/D(R)$. We shall identify the elements of the group F, as well as the normal divisors R and S, with their images in A. We know the following: A is generated by the elements a and b; the elements $a^p, a^{-i}ba^i$, $0 \le i \le p-1$, $\mathfrak{D}$-freely generate R; the normal divisor S lies in $\mathfrak{D}$ and contains the element $c = az = a^{pn+1}b^m w$, where $z \in R$, $w \in A'$.

Lemma 2. *In the group A, the elements $c^p, c^{-i}bc^i$, $0 \le i \le p-1$, multiply $\mathfrak{D}$-freely.*

This lemma is close to Lemma 5.9 of ([5]). One can prove that the elements under consideration satisfy the conditions of Lemma 2 in the $\mathfrak{D}$-free group R.

Corollary 1. In the group A, the elements $c^p, b, [c^i, b]$, $1 \le i \le p-1$, are $\mathfrak{D}$-freely multiplied.

It suffices to indicate that $[c^i, b] = (c^{-i}bc^i)^{-1}b$.

Lemma 3. Let the group H be $\mathfrak{D}$-freely generated by elements x and y. Then the elements x and $[x, y]$ also are $\mathfrak{D}$-freely multiplied.

Consider the homomorphism $\varphi : H \to S$, defined by $x\varphi = [c, b]$, $y\varphi = c$. As is not hard to see, it is enough for us to prove that the elements $[c, b]$ and

$$[[c, b], c] = [c, b]^{-1}[c^2, b][c, b]^{-1}$$

are $\mathfrak{D}$-freely multiplied. This is equivalent to saying that the elements $[c, b]$ and $[c^2, b]$ are $\mathfrak{D}$-freely multiplied. For $p > 2$ this follows directly from Corollary 1.

For $p = 2$ one should consider homomorphisms of the subgroup $U \subset A$, $\mathfrak{D}$-freely generated by the elements $c^2, b, [c, b]$, into the group S, given by the maps $[c, b] \to c^2$, $c^2 \to [c, b]$, $b \to c$; $[c, b] \to [c, b]^{-1}$, $c^2 \to c^2$, $b \to 1$, and $[c, b] \to c$, $c^2 \to [c, b]$, $b \to 1$. It is easy to see that under the successive application of these homomorphisms the images of the elements $[c, b]$ and $[c^2, b]$ lie in U all the time. After applying all three homomorphisms, we see that the elements $[c, b]$ and $[c^2, b]$ go to the elements $[c, b], c^2$, which, by Corollary 1, are $\mathfrak{D}$-freely multiplied. Hence the assertion of the lemma follows for $p = 2$.

Lemma 4. In the group A, the elements c and $[c, b]$ are $\mathfrak{D}$-freely multiplied.

Proof. Consider the images a^*, b^*, c^* of the elements a, b, c under the natural epimorphism $A \to A^* = A/D(A)$. It is easy to see that the elements a^*, b^* $\mathfrak{D}$-freely generate A^*, while the elements c^* and b^* satisfy the conditions of Lemma 1; then, by Lemmas 1 and 3, the elements c^* and $[c^*, b^*]$, to which the elements c and $[c, b]$ are mapped, are $\mathfrak{D}$-freely multiplied. This proves Lemma 4.

Further, the proof of the theorem is carried out analogously to the proof of Theorem 4.1 of the paper ([5]).

Consider the group H, $\mathfrak{D}$-freely generated by elements x and y, and in it the normal divisor $R_1 = \{x^p, y\}^H$. It, as a subgroup, is generated by the elements x^p, q_i, $0 \le i \le p-1$, where $q_i = x^{-i}yx^i$.

Lemma 5. In the group H the following holds: α) for each i, $0 \le i \le p-1$, the two elements x^p, q_i are $\mathfrak{D}$-freely multiplied; β) for each i, $0 \le i \le p-1$, there exists an endomorphism $\varepsilon_{1,i} : R_1 \to R_1$ such that

$$x^p \varepsilon_{1,i} = x^p, \quad q_i \varepsilon_{1,i} = q_i, \quad q_j \varepsilon_{1,i} = 1, \quad 0 \le j \le p-1, \quad i \ne j;$$

γ) the elements q_0 and q_{p-1} are $\mathfrak{D}$-freely multiplied.

Proof. In view of Lemma 5 it is enough to prove the corresponding assertions for the elements $c^p, z_i,\ 0 \le i \le p-1$, where $z_i = c^{-i}[c,b]c^i$. It is easy to see that these elements lie in the subgroup U, $\mathfrak{D}$-freely generated by the elements $c^p, c^{-i}bc^i,\ 0 \le i \le p-1$. For each i construct a homomorphism χ_i of the group U into the group H, $\mathfrak{D}$-freely generated by elements x and y, as follows:

$$c^p \to x, \quad c^{-k}bc^k \to y, \quad k = 0, 1, \dots, i,$$

$$c^{-m}bc^m \to x^{-1}yx, \quad m = i+1, \dots, p-1.$$

It is easy to see that $c^p\chi_i = x,\ z_i\chi_i = [x,y],\ z_j\chi_i = 1,\ i \ne j$. By Lemma 4, x and $[x,y]$ are $\mathfrak{D}$-freely multiplied. This proves α).

Let φ_i be the restriction of χ_i to $\{c^p, z_1, \dots, z_{p-1}\} = U$, and ψ_i to $\{c^p, z_i\}$. Then ψ_i is an isomorphism and $\operatorname{Im}\varphi_i = \operatorname{Im}\psi_i$. As is not hard to see, one may put $\varepsilon_{1,i} = \varphi_i\psi_i^{-1}$.

To prove γ), construct the following homomorphism $U \to H$:

$$c^p \to x, \quad b \to y, \quad c^{-i}bc^i \to 1, \quad 1 \le i \le p-1.$$

Under this homomorphism the elements z_0 and z_{p-1} go to the elements y and $x^{-1}y^{-1}x = [x,y]y^{-1}$, which are $\mathfrak{D}$-freely multiplied. Lemma 6 is proved.

Consider now in the group H the normal divisors $R_n = \{x^{p^n}, y\}^H$. Clearly, $R_n \supseteq R_{n-1}$; moreover, R_n, as a subgroup, is generated by the elements $x^{p^n}, q_i,\ 0 \le i \le p^n - 1$, where $q_i = x^{-i}yx^i$.

Lemma 6. In the group H the following holds: α^*) for each i, $0 \le i \le p^n - 1$, the two elements x^{p^n}, q_i are $\mathfrak{D}$-freely multiplied; β^*) for each i, $0 \le i \le p^n-1$, there exists an endomorphism $\varepsilon_{n,i} : R_n \to R_n$ such that,

that $x^{p^n}\varepsilon_{n,i} = x^{p^n}$, $q_i\varepsilon_{n,i} = q_i$, $q_j\varepsilon_{n,i} = 1$, $0 \le j \le p^n - 1$, $i \ne j$; γ^*) for all j, $0 \le j \le p^{n-1} - 1$, the two elements q_j and $q_{j+(p-1)p^{n-1}}$ multiply $\mathfrak{D}$-freely.

The proof is by induction on n and coincides word for word with the proof of Lemma 4.3 of paper (5).

Put

$$D_{n,j} = \{q_j,\ q_{j+(p-1)p^{n-1}}\}, \quad 0 \le j \le p^{n-1} - 1.$$

From Lemma 6 and the definition of the subgroup $D_{n,j}$ we obtain three corollaries:

Corollary 2. $D_{n,j}$ is isomorphic to H for all n, j, $0 \le j \le p^{n-1} - 1$.

Corollary 3. $D_{n,j} = x^{-j} D_{n,0} x^j$.

Corollary 4. For each n, the subgroups $D_{n,j}$ multiply properly (see (7)) over all j, $0 \le j \le p^{n-1} - 1$.

Corollaries 2 and 3 are trivial. To prove Corollary 4, we observe that $\varepsilon_{n-1,j}$ acts identically on $D_{n,j}$ and maps $D_{n,k}$ to E when $0 \le k \le p^{n-1} - 1$, $j \ne k$.

Lemma 7. Suppose a group Q generates the variety $\mathfrak{B}$. Suppose further that in some group K, for each n, there is a subgroup B_n, an element d_n, and natural numbers s_n such that: $Q \approx B_n$; all the subgroups $B_n, d_n^{-1} B_{nd} n, \dots, d_n^{-s_n} B_{nd} n^{s_n}$ are distinct and multiply properly, and the numbers s_n are unbounded in the aggregate. Then $\mathrm{var}(K) \supseteq \mathfrak{B} \circ \mathfrak{A}$, where $\mathfrak{A}$ is the variety of all abelian groups.

This lemma generalizes Lemma 2.2 of paper ([5]) and can be proved in exactly the same way.

Apply Lemma 7 to the group H, $\mathfrak{D}$-freely generated by the elements x and y. For each n put $B_n = D_{n,0}$, $d_n = x$, $s_n = p^{n-1} - 1$. Corollaries 2, 3, 4 show that we are in the conditions of the lemma. Then we obtain $\mathrm{var}(H) \supseteq \mathrm{var}(H) \circ \mathfrak{A}$, whence $\mathrm{var}(H) = \mathfrak{C}$, the variety of all groups. But the group H, by construction, lies in $\mathfrak{D}$, therefore $\mathfrak{D} = \mathfrak{C}$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.