UNATTAINABLE SUBGROUPS AND THE NORMAL STRUCTURE OF FINITE GROUPS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 519.44

MATHEMATICS

V. D. CHERTOK

UNATTAINABLE SUBGROUPS AND THE NORMAL STRUCTURE OF FINITE GROUPS

(Presented by Academician V. M. Glushkov on 31 V 1966)

§ 1. The presence in a group of unattainable subgroups and their mutual arrangement may exert a very substantial influence on the normal structure of the group. An example of this is the well-known theorem of Frobenius ([1]), according to which the existence in a finite group of a subgroup that coincides with its normalizer and is mutually prime with all its conjugate subgroups, i.e. is "very far" from a normal divisor, entails the existence in the group of an additional invariant subgroup. Moreover, as J. Thompson ([2]) proved, the additional invariant set is always nilpotent.

In § 3 finite groups with unattainable subgroups are considered and, under certain additional requirements, the following are proved: the existence of an invariant complement in the group (Theorems 3 and 4), and solvability of the group (Theorems 1, 2, 5, and 6). Theorem 7 gives a description of groups for which $\mathrm{ev}(G) = 4$ (see the definition below).

In § 4 the normal structure of a finite group is investigated in connection with the permutability of its subgroups.

Since groups in which all i-th ($i = 2, 3$) maximal subgroups are invariant are not generated by their i-th ($i = 2, 3$) maximal unattainable subgroups, Theorems 23 and 24 of B. Huppert ([4]), respectively, are special cases of Theorems 1 and 2; Theorem 2 generalizes Theorem 12 of Ya. G. Berkovich ([5]); Theorem 3 is an analogue of a theorem of R. Carter ([6]); Theorem 5 generalizes Herstein' s theorem ([7]) on the solvability of groups with a maximal abelian subgroup; from Theorem 7 follows the main result of V. Deskins ([3]); the corollary generalizes the known theorem of B. Huppert ([8]) on the supersolvability of the product of two permutable cyclic groups; Theorem 9 refines Theorem 5 of B. Huppert ([9]); Theorems 10 and 11 supplement the estimates of Hall–Higman ([10]) for the p-length of a p-solvable group (Theorem 1.26 and Theorem A).

§ 2. We shall use the following definitions and notation: G is a finite group of order (G); Π is some set of prime numbers; $(G)_\Pi$ is the greatest Π-divisor of (G); an S_Π-subgroup is a subgroup of order $(G)_\Pi$ of the group G; $\lambda(m)$ is the number of all, not necessarily distinct, prime divisors of the natural number m.

A group G is called Π-**decomposable** if it decomposes into a direct product of two sets, one of which is its nilpotent S_Π-subgroup ([11]).

Let P be some Sylow p-subgroup of the group G, and let

$$z_0 = 1 \subseteq z_1(P) \subseteq z_2(P) \subseteq \dots \subseteq z_n(P) = P$$

be the upper central series of the subgroup P.

A group G is called **strongly p-normal** if $z_i(P)$, $i = 1, 2, \dots, n-1$, is invariant in every Sylow p-subgroup containing it ([12]).

A finite group G is called, according to S. A. Chunikhin ([13]), p-solvable (p-supersolvable) if G has a composition (chief-

row), each index of which is either equal to the prime number p, or is not divisible by p.

Let

$$1 = P_0 \subseteq N_0 \subset P_1 \subset N_1 \subset \dots \subset N_l = G$$

be the upper invariant p-series of the p-solvable group G, i.e., such a series in which N_i/P_i $(i = 0, 1, \dots, l)$ is the largest invariant p'-subgroup in G/P_i, and P_i/N_{i-1} is the largest invariant p-subgroup in G/N_{i-1} $(i = 1, 2, \dots, l)$.

The minimal number l for which $N_l = G$ is called, following Hall and Higman [(10)], the p-length of the p-solvable group G.

A set of subgroups of a finite group G, $M_0 = G, M_1, \dots, M_n$, is called an upper even chain C_n of length n if each subgroup M_i is maximal in M_{i-1}, $i = 1, 2, \dots, n$, and has even order or is the identity subgroup.

Following V. Deskins [(3)], we introduce the concept of even variation for a group G. The ratio $n/s(C_n)$ ($s(C_n)$ is the number of subgroups $M_i \neq M_0$ of the upper even chain C_n that are attainable in G), if $s(C)_n \neq 0$, and n if $s(C_n) = 0$, is called the even variation of C_n. Among the even maximal chains of subgroups of the group G we choose one that has the greatest even variation; the variation of this chain is called the even variation of the group G and is denoted by $\mathrm{ev}(G)$.

On Schmidt groups, see [(14,15)]. On the groups $SL(2,5)$ and $LF(2,p^n)$, see [(16)] or [(17)].

We present the results obtained.

§ 3. **Theorem 1.** *Let G not be generated by a set of second maximal unattainable subgroups. Then G is solvable. If G is a non-nilpotent group, then it can be of two types:*

1) *$G = PQ$, where P is a cyclic p-group and Q is a minimal normal divisor of the group G.*

2) *$G = (PQ) \times R$, where P and R are cyclic subgroups, and PQ is a Schmidt group, with Q a minimal normal divisor of the group G.*

Theorem 2. *If G is not generated by a set of third maximal unattainable subgroups, then G is solvable.*

Theorem 3. *Let H be a nilpotent S_Π-subgroup that coincides with its normalizer in the group G. If G is strongly p-normal for all $p \in \Pi$, then H has an invariant complement in G.*

Theorem 4. *Let H be an S_Π-subgroup of the group G, and suppose that some maximal subgroup M of G containing H is Π-decomposable. If H has a Sylow p-subgroup whose center is not invariant in G, and if G is strongly p-normal, then G has an invariant complement to H.*

Theorem 5. *Let M be a maximal nilpotent subgroup of the finite group G. If G is strongly 2-normal, then it is solvable and its 2-length is equal to 1.*

Theorem 6. *A finite group of even order is solvable if* $\mathrm{ev}(G) < 4$.

Theorem 7. *Let G be a nonsolvable group with $\mathrm{ev}(G) = 4$. Then G is isomorphic to one of the following groups:*

1) *the special linear group $SL(2,5)$;*
2) *the fractional-linear group $LF(2,p^n)$, except that: a) if 4 divides $(p-1)$, then $\lambda(p-1) \leq 3$, $n = 1$; b) if 4 divides $(p+1)$, then $\lambda(p^n+1) \leq 3$, $p \neq 7$, $\lambda(p^n-1) \leq 3$.*

Theorem 6 weakens the condition of theorem 3 of [3] for groups of even order, and theorem 7 weakens the condition of theorem 17 of [18].

§ 4. **Theorem 8.** *Let the p-solvable group G be representable in the form $G = AB$, where A and B are p-decomposable groups with cyclic Sylow p-subgroups. Then G is p-supersolvable.*

In the theorem just given, the condition of p-decomposability of the factors is essential, as is seen from the example of the octahedral group, which is not 2-supersolvable, although it is representable as a product of subgroups (6.4) with cyclic Sylow p-subgroups.

Since the product of permutable nilpotent subgroups is soluble [19], Theorem 8 implies

Corollary. Let $G = AB$, where A and B are nilpotent groups, and let the Sylow p-subgroups of A and B be cyclic. Then G is p-supersolvable.

If the condition of the corollary is satisfied for all prime divisors of the order of the group, then we obtain a result of B. Huppert: a finite group admitting a factorization by two cyclic subgroups is supersolvable [8].

Theorem 9. Let P be a Sylow p-subgroup and let Q be a p-complement in G; suppose that $(P) > p$ and every maximal subgroup of P is permutable with Q.

Then G is a p-supersolvable group.

Let P be a Sylow p-subgroup of the group G, and let

$$P = \Gamma_0(P) \supseteq \Gamma_1(P) \supseteq \cdots \supseteq \Gamma_n(P) = 1$$

be its lower central series.

Theorem 10. Let G be a p-solvable group with Sylow p-subgroup P and p-complement Q. If the k-th term $(1 \le k \le n)$ of the lower central series of the Sylow p-subgroup P is permutable with Q, then $l_p(G) \le k$.

From Theorem 10, when $k = 1$, there follows one result of [20].

Theorem 11. Let G be a p-solvable group with Sylow p-subgroup P and p-complement Q. If the k-th term $(1 \le k \le n)$ of the commutator series of the Sylow p-subgroup P is permutable with Q and $p > 2$, then $l_p(G) \le k$.

Theorem 12. Let G be a p-solvable group with Sylow p-subgroup P and p-complement Q. If the first k, distinct from E, terms of the upper central series of the Sylow p-subgroup P are permutable with G, then $l_p(G) \le c_p - k$.

I express my deep gratitude to Prof. S. A. Chunikhin for his attention to the work, and also to V. I. Sergienko for valuable advice.

Institute of Mathematics
Academy of Sciences of the BSSR

Received
22 IV 1966

CITED LITERATURE

1. G. Frobenius, Berlin. Sitzber., 1223 (1901).
2. J. G. Thompson, Proc. Nat. Acad. Sci. U.S.A., 45, 578 (1959).
3. W. E. Deskins, Illinois J. Math., 5, No. 2, 306 (1961).
4. B. Huppert, Math. Zs., 60, 409 (1954).
5. Ya. G. Berkovich, DAN, 169, No. 3, 499 (1966).
6. R. W. Carter, Math. Zs., 78, 149 (1962).
7. I. Herstein, Proc. Am. Math. Soc., 9, 255 (1958).
8. B. Huppert, Math. Zs., 58, 243 (1953).

9. B. Huppert, Arch. Math., 12, No. 3, 161 (1961).

10. P. Hall, G. Higman, Proc. London Math. Soc., Ser. (3), 6, 1 (1956).

11. I. K. Chunikhin, S. A. Chunikhin, Matem. sborn., 15(57), 325 (1944).

12. S. Bauman, Math. Zs., 87, No. 2, 345 (1965).

13. S. A. Chunikhin, DAN, 55, No. 6, 481 (1947).

14. O. Yu. Schmidt, Matem. sborn., 31, 366 (1924).

15. S. A. Chunikhin, Tr. seminara po teorii grupp, M.–L., 1938, p. 106.

16. L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, Leipzig, 1901.

17. W. Burnside, *Theory of Groups of Finite Order*, Cambridge, 1911.

18. Ya. G. Berkovich, Izv. AN SSSR, ser. matem., 28, 3, 583 (1964).

19. O. Kegel, Arch. Math., 12, 90 (1961).

20. N. Inagaki, Nagoya Math. J., 25, 205 (1965).

Note: Figure translations are in progress. See original paper for figures.