GROUP ALGEBRAS OF COUNTABLE ABELIAN p-GROUPS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 519-45

MATHEMATICS

S. D. BERMAN

GROUP ALGEBRAS OF COUNTABLE ABELIAN p-GROUPS

(Presented by Academician A. I. Mal' tsev, 29 IX 1966)

In the present note the problem of the isomorphism of group algebras of countable abelian p-groups is solved. The theorems formulated here were presented in the first part of the author' s report at the International Congress of Mathematicians (Moscow). Theorems 5 and 6 and one of the assertions of Theorem 2 have been partially published in ([1,2]).

We shall agree to denote by GK the group algebra of a group G over a field K. Deskins showed ([3]) that from the isomorphism of the group algebras GK and G_1K of two finite abelian p-groups G and G_1 over a field of characteristic p there follows the isomorphism of the groups G and G_1. Necessary and sufficient conditions for the isomorphism of the group algebras GK and G_1K of finite p-groups G and G_1 over a field K whose characteristic is different from p were found in ([4,5]).

Theorem 1. *The group algebras GK and G_1K of two countable abelian p-groups G and G_1 over a field K of characteristic p are isomorphic if and only if the groups G and G_1 are isomorphic.*

Let now p be a prime number, and let K be a field whose characteristic is different from p. Denote by ξ_i a primitive root of unity of degree p^i $(i = 1, 2, ...)$.

We shall call a field K a field of the first kind relative to the prime p if $K(\xi_j) \neq K(\xi_2)$ for some $j > 2$. Otherwise we shall call K a field of the second kind (relative to the prime p).

Theorem 2. *Let K be a field of the first kind relative to the prime p. Every countable primary abelian p-group G belongs to exactly one of the following 9 types:*

1) G is a direct product of cyclic groups with element orders unbounded in the aggregate;
2) G is a direct product of cyclic groups with element orders bounded in the aggregate;
3) G is the group p^∞;

4) G is a complete group whose direct decomposition contains at least two groups p^∞;

5) G is the direct product of the group p^∞ by a finite p-group H $(H \neq 1)$;

6) G is the direct product of a complete group of type 4) by a finite p-group H $(H \neq 1)$;

7) G is the direct product of a complete group by an infinite p-group without elements of infinite height and with element orders bounded in the aggregate;

8) G is a reduced p-group, and the subgroup P of elements of infinite height in G is finite and different from 1;

9) the subgroup P of elements of infinite height in G is infinite, and the orders of elements of the factor group G/P are not bounded in the aggregate.

If abelian p-groups G and G_1 belong to different types, then the group algebras GK and G_1K are not isomorphic.

If G and G_1 are groups of one and the same type $n = 1, 3, 4, 8, 9$, then the group algebras GK and G_1K are isomorphic.

Let G and G_1 be groups of type 2), $p^\alpha(p^{\alpha_1})$ the exponent of the group $G(G_1)$, and $p^\beta(p^{\beta_1})$ the greatest of the orders of those cyclic direct factors of the group $G(G_1)$ that occur in a direct decomposition of the group $G(G_1)$ a countable number of times. Let $\eta(\eta_1)$ be a primitive $p^\alpha(p^{\alpha_1})$-th root of unity, and $\varepsilon(\varepsilon_1)$ a primitive $p^\beta(p^{\beta_1})$-th root of unity. The group algebras GK and G_1K are isomorphic if and only if $(K(\eta) : K) = (K(\eta_1) : K)$, $(K(\varepsilon) : K) = (K(\varepsilon_1) : K)$.

Let G and G_1 be simultaneously groups of type 5) or 6): $G = P \times H$, $G_1 = P_1 \times H_1$ (P and P_1 are groups p^∞ or complete groups of type 4), H and H_1 are finite p-groups). The algebras GK and G_1K are isomorphic if and only if the algebras HK and H_1K are isomorphic.

Finally, let G and G_1 be groups of type 7): $G = P \times H$, $G_1 = P_1 \times H_1$ (P and P_1 are complete groups, H and H_1 are countable p-groups of type 2)). The algebras GK and G_1K are isomorphic if and only if the algebras HK and H_1K are isomorphic.

Theorem 3. If $p \neq 2$, and K is a field of the second kind (relative to the prime p), then the group algebras GK and G_1K of any two countable abelian p-groups G and G_1 are isomorphic.

Theorem 4. Let $p = 2$, and let K be a field of the second kind relative to p. If $K = K(\xi_2)$ (ξ_2 is a primitive fourth root of unity), then $GK \simeq G_1K$ for any two countable abelian 2-groups G and G_1.

If $K \neq K(\xi_2)$, then the group algebra GK of an arbitrary countable abelian 2-group G is isomorphic to the group algebra of a 2-group of one of the following types: $G_1 = (2, \dots, 2, \dots)$ (the direct product of a countable number of cyclic groups of order 2); $G_2 = (4, 2, \ldots, 2, \ldots)$; $G_3 = (4, \ldots, 4, \ldots)$; G_4 is the group 2^∞; $G_5^s = G_4 \times H_s$, where G_4 is the group 2^∞, $H_s = (2, \ldots, 2)$ is the direct product of s cyclic groups of order 2 ($s \geq 1$).

The group algebras of 2-groups of different types are not isomorphic.

1) $GK \simeq G_1K$ if and only if $G \simeq G_1$;
2) $GK \simeq G_2K$ if and only if G contains no elements of infinite height and, in the decomposition of the group G into a direct product of cyclic groups, only a finite number of factors occur whose orders are greater than 2.
3) Let P be the subgroup of elements of infinite height of the group G. $GK \simeq G_3K$ if and only if, in the decomposition of the group G/P into a direct product of cyclic 2-groups, infinitely many cyclic groups occur whose orders are greater than 2.
4) $GK \simeq G_4K$ if and only if G is a complete 2-group.
5) $GK \simeq G_5^s$ if and only if $G = P \times H$, where P is a complete 2-group, and H is a finite 2-group decomposing into a direct product of s cyclic groups.

Theorem 5. Let G and G_1 be countable periodic abelian groups, and let K be an algebraically closed field whose characteristic does not divide the orders of the elements of the groups G and G_1. Then the group algebras GK and G_1K are isomorphic.

Theorem 6. The group algebra GD of an arbitrary countable periodic abelian group G over the field of real numbers D is isomorphic to the real group algebra of one of the 2-groups listed in the formulation of Theorem 4. Represent the group G in the form of a direct product $G = N \times P \times R$, where N is the subgroup of elements of odd order, P is a complete 2-group, and R is a reduced 2-group. If the subgroup $N \times P$ is infinite and the group R is finite, then $GD \simeq G_5^{sD}$, where s is the number of cyclic direct factors in the decomposition of the group R.

$GD \simeq G_3D$ if and only if the subgroup R is infinite and at least one of the following conditions is satisfied:

1) $N \times P$ is an infinite group;
2) the subgroup R contains elements of infinite height;
3) $N \times P$ is a finite group, and the group R decomposes into a direct product of cyclic groups, among which there are infinitely many groups whose orders are greater than 2.

$GD \cong G_2D$ if and only if the group R decomposes into a direct product of a countable number of cyclic groups, R^2 and $N \times P$ are finite groups, and $R \ncong G$ if $N \times P = 1$.

$GD \cong G_1D$ if and only if $G \cong G_1$.

Theorem 7. Let G be a periodic group (of arbitrary cardinality), and let K be a field of characteristic zero. Every indecomposable GK-module is irreducible. The irreducible representations of the group G over the field K are in one-to-one correspondence with such sets E of idempotents of the algebra GK that:

1) the elements of E are the minimal idempotents of the group subalgebras HK of all possible finite subgroups H of the group G; for each finite subgroup $H \subseteq G$, the set E contains exactly one minimal idempotent $e \in HK$;

2) any two idempotents from the set E are not orthogonal; irreducible representations Γ and Γ_1 of the group G over the field K are equivalent if and only if the corresponding sets of idempotents E and E_1 coincide.

Theorem 8. Let G and G_1 be countable abelian p-groups; K a countable or finite field of characteristic p; S and S_1 the Sylow p-subgroups of the group multiplicative groups, respectively, of the algebras GK and G_1K; $P(P_1)$ the maximal complete subgroup of the group $G(G_1)$; $w(w_1)$ the ordinal type of the Ulm series of the group G/P (G_1/P_1). If $w = \gamma + 1$ $(w_1 = \gamma_1 + 1)$ is a transfinite number of the first kind, then denote by $G^{\gamma}(G_1^{\gamma_1})$ the last factor of the Ulm series of the group G/P (G_1/P_1). In this case the group $G^{\gamma}(G_1^{\gamma_1})$ decomposes into a direct product of cyclic groups. Let $p^{\beta}(p^{\beta_1})$ be the greatest of the orders of those cyclic direct factors of the group $G^{\gamma}(G_1^{\gamma_1})$ which occur in the decomposition of the group $G^{\gamma}(G_1^{\gamma_1})$ a countable number of times; $H(H_1)$ the direct product of all cyclic direct factors of the group $G^{\gamma}(G_1^{\gamma_1})$ whose orders do not exceed $p^{\beta}(p^{\beta_1})$.

The groups S and S_1 are isomorphic if and only if the following conditions are simultaneously satisfied:

1) if $P \neq 1$, then $P_1 \neq 1$;

2) $w = w_1$;

3) if $w = w_1 = \gamma + 1$ and K is a countable field, then the groups G^{γ} and G_1^{γ} have one and the same exponent p^{α}, or the orders of the elements of these groups are unbounded. If $w = \gamma + 1 = w_1$ and K is a finite field, then the orders of the elements of the groups G^{γ} and G_1^{γ} are simultaneously unbounded or bounded; moreover, in the latter case the exponents of the groups G^{γ} and G_1^{γ} coincide, and the finite groups G^{γ}/H and G_1^{γ}/H_1 are isomorphic.

Uzhgorod State
University

Received
19 IX 1966

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Note: Figure translations are in progress. See original paper for figures.