ON FIELDS WITH A SOLVABLE THEORY

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Abstract

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In the present note some new examples are indicated of fields and classes of fields of finite characteristic having a decidable elementary theory (for definitions connected with mathematical logic, see [1]).

1. Let p be a prime number, and let F be a field of characteristic p. Put

$$\tau(F) = \begin{cases} \log_p [F : F^p], & \text{if } [F : F^p] \text{ is finite,} \\ \infty, & \text{if } [F : F^p] \text{ is infinite.} \end{cases}$$

A field F is **separably closed** if it has no proper algebraic separable extensions.

Proposition. *Let F_1 and F_2 be separably closed fields of characteristic $p \neq 0$. F_1 and F_2 are elementarily equivalent if and only if $\tau(F_1) = \tau(F_2)$.*

We shall give the proof only for the case $\tau(F_1) = \tau(F_2) = \tau < \infty$. Let T be a system of axioms in the signature of the theory of fields with added symbols for constants $a_1, \dots, a_\tau$, satisfied only by models of the form $\langle F, a_1, \dots, a_\tau \rangle$, where F is a separably closed field of characteristic p, $\tau(F) = \tau$, and $a_1, \dots, a_\tau$ form a p-basis of F over F^p [5]. We shall show that the theory T is complete.

First, T is model complete. Indeed, let $\langle F', a_1, \dots, a_\tau \rangle \subset \langle F'', a_1, \dots, a_\tau \rangle$ be two models of T. Then F'' is a regular extension of F' [2]. In fact, F'' is separable over F', since $a_1, \dots, a_\tau$ remains a p-basis in F'' over F''^p; moreover, F' is separably closed and, consequently, algebraically closed in F''. From the regularity of F'' over F' and the separable closedness of F' it follows [2] that every point of F'' has a specialization in F' over F'. Hence the model completeness of T follows at once.

Second, among the models of the theory T there is a minimal one. Namely, consider the field $Z_p(a_1, \dots, a_\tau)$ of rational functions in τ variables over the prime field Z_p. The separable closure of this field is the minimal model.

From Robinson' s criterion [1] follows the completeness of the theory T. This completes the proof in the case under consideration. The case $\tau(F_1) = \infty$ is considered somewhat more complicatedly.

Theorem 1. *The following classes of fields have a decidable theory:*

a) *the class of all separably closed fields;*

b) *the class of all separably closed fields of fixed characteristic;*

c) *the class consisting of one (arbitrary) separably closed field.*

Theorem 1 may be regarded as a natural extension of A. Tarski' s result on the decidability of the theory of algebraically closed fields.

2. A finite field with p^r elements, where p is a prime number, will be denoted by $F_{p,r}$. Let F be an absolutely algebraic field of characteristic p,

i.e., the algebraic extension of the field Z_p. With the field F we associate the following function χ_F:

$$\chi_F(n) = \begin{cases} \max\limits_s \left(F_{p,p_n^s} \subset F\right) + 1, & \text{if there exists an } s \text{ such that } F_{p,p_n^s} \not\subset F, \\ 0, & \text{otherwise,} \end{cases}$$

where p_n is the n-th prime number.

Let us note that for any two absolutely algebraic fields F' and F'' of characteristic p,

$$F' = F'' \Longleftrightarrow \chi_{F'} = \chi_{F''},$$

and for any function χ defined on $\{0, 1, ...\}$ and taking values in the same set, there exists an absolutely algebraic field F of characteristic p such that $\chi = \chi_F$.

Theorem 2. Let F be an absolutely algebraic field of characteristic $p \neq 0$. Then F has a decidable theory if and only if χ_F is a recursive function.

Theorem 3. Let $F_1, F_2, ...$ be a sequence of finite fields such that: 1) $F_i \subset F_{i+1}$; 2) the set of cardinalities $\{\overline{F_i}\}$ is recursive; 3) if $F_\infty = \bigcup F_i$, then χ_{F_∞} is a recursive function not taking the value 0. Then the class $\{F_1, F_2, ...\}$ has a decidable theory.

The proofs of Theorems 2 and 3 use the technique of model completeness and an important result of Lang and Weil (3).

For the proof, one writes down a recursive system of axioms which assert that, if a system of polynomials defines an absolutely irreducible variety (see (4)) and the field contains sufficiently many elements (the estimate from (3) is used), then this variety has a rational point.

The second group of axioms consists of axioms asserting the existence or nonexistence of subfields of the form F_{p,p_n^s} (the existence or nonexistence of subfields of this kind in the presence of a sufficient number of elements), and here the function $\chi_F(\chi_{F_\infty})$ is used.

The third group of axioms describes the Galois group of the algebraic closure of the field F (asserts that the Galois group of the closure of the field is $\hat{Z}$ in Theorem 3).

In the proof of Theorem 3 there is also a fourth group of axioms, by means of which the finite fields different from $F_1, F_2, \ldots$ are excluded.

For the constructed systems of axioms it is proved that they are systems of axioms for the theory of the field F in Theorem 2 and for the theory of the class $\{F_1, F_2, \ldots\}$ in Theorem 3, whence, from the recursiveness of these systems of axioms (in the case of recursiveness of χ_F in Theorem 2 or of the fulfillment of all the hypotheses of Theorem 3), decidability follows. The undecidability of the theory of the field F in the case of nonrecursiveness of χ_F in Theorem 2 is obvious.

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Note: Figure translations are in progress. See original paper for figures.