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Abstract

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ON UNIFORMLY NONELLIPTIC SYSTEMS OF SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS ON COMPACT MANIFOLDS

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1. Let X be a smooth n-dimensional compact manifold and let S be a square matrix of singular integro-differential (s.i.-d.) operators on X. At present the properties of the system $Su = f$ have been well studied in the case when the matrix S is elliptic. In particular, such a system defines a Noetherian operator from $H^s(X)$ to $H^{s-m}(X)$, where m is the order of the system S (i.e. S defines a continuous operator whose kernel and cokernel are finite-dimensional). (For the literature on this question see ([1–3]).)

In the present note a certain class of nonelliptic systems is singled out which nevertheless are Noetherian in some other spaces. These spaces will be defined below. In the second part of the work the case $n = 1$ (X a curve), which has certain specific features, is considered in greater detail.

We first recall the definition of an s.i.-d. operator. An operator P on X is called an s.i.-d. operator of order not exceeding m if:

1) for any function $\varphi(x)$ equal to zero on some open set $U \subset X$, the function $P\varphi$ is infinitely differentiable on the set U. 2) Let U be any sufficiently small neighborhood on X and let $x = (x_1, \dots, x_n)$ be local coordinates on U. Then for any $N > 0$ there exist functions $K_N(x, \xi)$, infinitely differentiable with respect to x, ξ and, for $|\xi| > 1$, of the form

$$K_N(x, \xi) = \sum_{j=0}^{N+m-1} a_j(x, \xi), \tag{1}$$

where $a_j(x, \xi)$ are homogeneous functions of ξ of order $m - j$, such that for any function $\varphi(x)$ with compact support in U

$$P\varphi\big|_U = F^{-1} K_{NF}\varphi + T_N, \tag{2}$$

where F is the Fourier transform in R^n_x; F^{-1} is the inverse Fourier transform; T_N is a smoothing operator increasing smoothness by N units (as $\varphi(x)$ one may consider arbitrary generalized functions on X).

The function $a_0(x,\xi)$ is called the **symbol of the operator** P and is denoted by $\sigma(P)$. Note that the symbol $\sigma(P)$ is a function on $T^*_0(X)$, where $T^*_0(X)$ is the part of the cotangent bundle $T^*(X)$ consisting of nonzero vectors. It is possible for the symbol to be identically equal to zero.

We shall consider a matrix S of s.i.-d. operators of order not exceeding m. In this case we shall say that the matrix S has **order** m. By the symbol $\sigma(S)=\sigma(x,\xi)$ of the matrix S we shall mean the matrix composed of the symbols of the operators $S_{i,j}$. In what follows we shall also encounter rectangular matrices of s.i.-d. operators of order not exceeding m, and the symbol of such matrices is likewise defined as the matrix of symbols.

2. Thus, let S be a square $t\times t$ matrix of s.i.-d. operators of order not exceeding m. If the matrix $L_0=\sigma(S)$ has maximal rank at every point of $T^*_0(X)$, then the system of equations

$$Su=f \tag{3}$$

is called **elliptic**. Suppose that at every point of $T^*_0(X)$ the matrix L_0 has one and the same rank, equal to r_0. Let Λ_0 be some rectangular matrix on $T^*_0(X)$, of first order of homogeneity in ξ, depending smoothly on the points of $T^*_0(X)$, having rank $t-r_0$, and such that $\Lambda_0 L_0\equiv 0$ (this matrix specifies all relations between the rows of the matrix L_0). Such a matrix Λ_0 always exists. By C_0 denote any, but fixed in some way, s.i.-d. system of order 1 with symbol Λ_0. Consider the matrix

$$L_1=\begin{pmatrix} L \\ \sigma(C_0 S)\end{pmatrix}.$$

Suppose that at all points of $T^*_0(X)$ the matrix L_1 has constant rank, equal to r_1. It is obvious that $r_1\geq r_0$. By Λ_1 denote a matrix depending smoothly on the points of $T^*_0(X)$ and of first order of homogeneity in ξ, specifying all relations between the rows of the matrix L_1; by C_1, an s.i.-d. system with symbol Λ_1. This process can now be repeated once more, taking together with L_1 the matrix

$$L_2=\begin{pmatrix} L_0 \\ \sigma\left(C_1\begin{pmatrix} S \\ C_0 S\end{pmatrix}\right)\end{pmatrix},$$

if, of course, L_2 has constant rank r_2, and so on.

Definition. The system (3) is called **uniformly nonelliptic** if, by the method indicated above, after a finite number of steps one obtains a matrix L_p of rank t at every point of $T_0^*(X)$.

Denote by $H_S^s(X)$ the space of functions $f = (f_1, \dots, f_t)$ for which

$$f,\ C_0 f,\ C_1 \begin{pmatrix} f \\ C_0 f \end{pmatrix}, \dots,\ C_p \begin{pmatrix} f \\ \cdots \\ C_{p-1} f \end{pmatrix}$$

belong to $H^{s-m}(X)$.

Lemma. The concept of uniform nonellipticity of the system (3), the numbers r_k, and the space $H_S^s(X)$ do not depend on the arbitrary choice of the matrices Λ_k and the operators C_k.

Theorem 1. If the system (3) is uniformly nonelliptic, then it defines a Noetherian operator S from $H^s(X)$ to $H_S^s(X)$.

The kernel of the operator S consists of infinitely differentiable functions, and the range is determined by conditions of the type of orthogonality to certain infinitely differentiable functions.

An obvious consequence of Theorem 1 is the following assertion.

Theorem 2. If the system (3) is uniformly nonelliptic, then for any function $f \in H^{s-m+p}(X)$ satisfying a finite number of conditions of the type of orthogonality to certain infinitely differentiable functions, there exists a solution u in the space $H^s(X)$. The homogeneous problem has a finite number of linearly independent solutions, and these solutions are infinitely differentiable.

The theorems formulated above are valid for any n, but for $n = 1$ one can obtain more precise assertions. These refinements are connected with the fact that in the case $n = 1$, in addition to s.i.-d. operators of order not exceeding m, one can also introduce s.i.-d. operators of order not exceeding r, q. Their definition differs in that, for an s.i.-d. operator of order not exceeding r, q, the functions

K_N, which occurs in formula (2), has for $|\xi| \geqslant 1$ the form

$$K_N(x,\xi) = \begin{cases} a_0(x)\xi^r + a_1(x)\xi^{r-1} + \dots + a_{N+r-1}(x)\xi^{-N+1}, & \xi > 0, \\ b_0(x)|\xi|^q + b_1(x)|\xi|^{q-1} + \dots + b_{N+q-1}(x)|\xi|^{-N+1}, & \xi < 0, \end{cases}$$

where x is a local parameter on the curve X, and ξ is the dual variable. The function $\sigma(P) = a_0(x)\xi^r$ for $\xi > 0$ and $\sigma(P) = b_0(x)|\xi|^q$ for $\xi < 0$ is called the symbol of the s.i.-d. operator P. The symbol $\sigma(P)$ is a function on $T_0^*(X)$. We shall say that $\sigma(P) \neq 0$ if $a_0(x) \neq 0$ and $b_0(x) \neq 0$.

For such operators one can prove all the usual theorems that hold for ordinary s.i.-d. operators: on the composition of operators, on passage to the adjoint operator, on change of variables, and on inversion of an operator with nonzero symbol. In particular, the following Noether–Muskhelishvili-type formula holds for the index $\varkappa(P)$ of such operators:

$$\varkappa(P) = \frac{1}{2\pi i}\left[\ln\frac{a_0(x)}{b_0(x)}\right]_X, \tag{4}$$

where $[f(x)]_X$ denotes the increment of the function $f(x)$ along the curve X.

The symbol of a system of s.i.-d. operators of order not exceeding r, q, as before, is called the matrix of the symbols of the corresponding operators. The notion of uniform nonellipticity of such systems is introduced in the same way as before, but it is now allowed that the matrix L_0 have different ranks for $\xi > 0$ and $\xi < 0$. Let these be respectively r_0^+ and r_0^-. It is assumed that r_0^+ and r_0^- do not depend on the choice of a point in X. As before, the matrix Λ_0 is constructed, only now it will have different ranks for $\xi > 0$ and $\xi < 0$. We proceed analogously further. The system is called uniformly nonelliptic if, after a finite number of steps, one obtains a matrix L_p having rank t both for $\xi > 0$ and for $\xi < 0$. For such systems, assertions analogous to Theorems 1 and 2 are valid.

Remark 1. If the manifold X is not connected, then it is allowed that the matrices L_k have different ranks on different components of the manifold $T_0^*(X)$.

Remark 2. All results remain valid if spaces of the type $H^s(X)$ are replaced by spaces of the type $C^{s,\alpha}(X)$, $s \geqslant m - p$.

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Note: Figure translations are in progress. See original paper for figures.