ON FINITELY MULTIPLE MAPPINGS

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Abstract

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In dimension theory the following two classical theorems of Hurewicz are known.

Theorem A (on dimension-lowering mappings). *If f is a closed* mapping of a metric space X with a countable base onto a metric space with a countable base Y, then* $\dim X \leq \dim Y + \dim f$, *where* $\dim f = \sup_{y \in Y}\{\dim f^{-1}y\}$ ***.*

Theorem B (on dimension-raising mappings). *If there exists a closed mapping of multiplicity*** $\leq k+1$ of a metric space X with a countable base onto a metric space with a countable base Y, then* $\dim Y \leq \dim X + k$.

Various authors have proposed generalizations of Theorem A (see ($^{1-4}$)) and of Theorem B (see (^{3,5,6})), mainly by replacing $\dim Y$ (respectively $\dim X$) by the dimensions $\operatorname{Ind} Y$ or $\operatorname{ind} Y$ (respectively $\operatorname{Ind} X$ or $\operatorname{ind} X$) in the cases under consideration. The most general of the precise forms of Theorem A for paracompact spaces was obtained not long ago by E. G. Sklyarenko with the aid of sheaf theory and the Leray spectral sequence.

An essential part of the present note is the proof of Theorems 2 and 3, which generalize and sharpen Theorem B. The proof uses a certain spectral sequence (Theorem 1).

The note also gives two theorems on open-closed finitely multiple mappings of cohomological manifolds, analogous to the theorems of A. V. Chernavskii for open-closed finitely multiple mappings of manifolds (^{8}). Their proof uses the construction of a homomorphism σ coinciding with the transfer homomorphism for finite groups of transformations.

In what follows we use the results, terminology, and notation of (^{9,10}).

Definition 1. We shall say that $\dim_A X \leq n$, where $\mathcal{A}$ is a sheaf over X, if $H^i(X; \mathcal{A}_U) = 0$ for $i > n$ and for every open $U \subseteq X$. For a subspace $M \subseteq X$, $\dim_A M \leq n$ if $\dim_{A|M} M \leq n$.

It is known that if X is paracompact and has finite dimension dim, then $\dim_Z X = \dim X$, where $\mathbb{Z}$ is the simple sheaf generated by the group of integers $\mathbb{Z}$ (the fundamental theorem of homological dimension theory). This shows the naturalness of Definition 1. The function $\dim_A X$ has additive properties analogous to those of the dimension dim: the union of a countable number of closed sets of dimension with respect to $\mathcal{A} \leq n$ itself has dimension with respect to $\mathcal{A} \leq n$; the union of a locally finite system of closed sets of dimension with respect to $\mathcal{A} \leq n$ itself has dimension with respect to $\mathcal{A} \leq n$, etc. Moreover, $\dim_A F \leq \dim_A X$ for every closed $F \subseteq X$. These and other useful facts are easily derived from the following lemma.

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* A mapping is called closed if the image of every closed set is a closed set.
** $\dim X$ is the dimension defined by means of the multiplicity of open covers.
*** The multiplicity of f is the greatest number of points in the preimages $f^{-1}y$.

Lemma 1. *In order that* $\dim_A X \leq n$, *it is necessary and sufficient that the sheaf* $\mathcal{A}$ *have a resolution of the form* $0 \to \mathcal{A} \to \mathcal{L}^0 \to \cdots \to \mathcal{L}^n \to 0$, *in which all the sheaves* $\mathcal{L}^i$ *are soft.*

Definition 2. We shall say that on the space X there is given a **partial inductive system of sheaves** $\Sigma = \{U_\lambda, \mathcal{A}_\lambda, \gamma_\mu^\lambda\}$, if: 1) a covering of X by open sets $\{U_\lambda\}$ and sheaves $\mathcal{A}_\lambda$ on U_λ are given; 2) in the set of indices $\Lambda = \{\lambda\}$ an order relation is introduced: if $\lambda \geq \mu$, then $U_\lambda \cap U_\mu = \emptyset$ and there exists a homomorphism of sheaves $\gamma_\mu^\lambda : \mathcal{A}_\lambda | U_\lambda \cap U_\mu \to \mathcal{A}_\mu | U_\lambda \cap U_\mu$, and, if $\lambda \geq \mu \geq \nu$, then $U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset$ and $\gamma_\nu^\lambda = \gamma_\nu^\mu \circ \gamma_\mu^\lambda$ on $U_\lambda \cap U_\mu \cap U_\nu$; 3) the set Σ is directed in the following sense: for every point $x \in U_{\lambda_1} \cap U_{\lambda_2}$ there exists an index $\mu \geq \lambda_1, \lambda_2$ such that $x \in U_\mu \cap U_{\lambda_1} \cap U_{\lambda_2}$. A partial inductive system is called regular if the homomorphisms γ_μ^λ are monomorphisms and if for every pair λ_1, λ_2 for which $U_{\lambda_1} \cap U_{\lambda_2} \neq \emptyset$ there exists $\mu \geq \lambda_1, \lambda_2$ such that $U_\mu = U_{\lambda_1} \cap U_{\lambda_2}$.

With a partial inductive system of sheaves Σ there is associated a sheaf—the limit of this partial inductive system $\operatorname{limp.ind} \Sigma$. Obviously, limp. ind may be regarded as a functor on the category of partial inductive systems with one and the same covering $\{U_\lambda\}$ and set of indices Λ. This functor has properties analogous to those of the inductive limit; for example, it is exact and commutes with the tensor product.

For example, let us note that every sheaf is the limit of a partial inductive system of constant sheaves, and every sheaf $\mathcal{A}$ is the limit of a regular partial inductive system of its subsheaves $\mathcal{A}_{U_\lambda}$ for any base $\{U_\lambda\}$, with identity mappings as the homomorphisms γ_μ^λ and the usual order.

Lemma 2. *The limit of a regular partial inductive system of relatively soft sheaves is a soft sheaf, if the space* X *is paracompact.*

From Lemmas 1, 2 and the mentioned exactness of the functor limp. ind it follows that

Lemma 3. $\dim_A X \leq n$, *if the sheaf* $\mathcal{A}$ *is the limit of a regular partial inductive system of sheaves whose relative dimensions are* $\leq n$.

Let now $f : X \to Y$ be a closed finite-fold mapping of a paracompact space X

onto a paracompact space Y; L a commutative ring with identity. For an open $U \subset Y$ and a finite partition $\sigma = \{U_i\}$ of its inverse image $f^{-1}U$ into pairwise disjoint open sets U_i, denote by

$$G^p(U,\sigma) = \prod_{[i_0,\dots,i_p]} \widetilde{G}_{[i_0,\dots,i_p]}. \tag{1}$$

Here

$$\widetilde{G}_{[i_0,\dots,i_p]} = f_* L_{[i_0,\dots,i_p]} \otimes_L \cdots \otimes_L f_* L_{[i_0,\dots,i_p]} \mid u, \quad L_{[i_0,\dots,i_j,\dots,i_p]} =$$

$$= LA_{[i_0,\dots,i_j,\dots,i_p]}, \quad \text{where } A_{[i_0,\dots,i_j,\dots,i_p]} = U_{i_j} \cap f^{-1}\left(\bigcup_{r=0}^{p} fU_{i_r}\right);$$

the product is taken over all ordered sets $[i_0,\dots,i_p]$. The sheaves $G^p(U,\sigma)$ naturally form a regular partial inductive system; its limit will be denoted by G^p. Moreover, in the graded sheaves $G^*(U,\sigma)$ there are defined differentials of degree $+1$; the limit of these differentials defines a differential in the graded sheaf G^*.

We now define sheaves $\widetilde{G^p} \subseteq G^p$. $\widetilde{G^p}(U,\sigma)$ is defined by the same formula (1), but this time the products are taken only over those sets $\{i_0,\dots,i_p\}$ for which $i_j \neq i_{j'}$ if $j \neq j'$; the order $[i_0,\dots,i_p]$ in each such set $\{i_0,\dots,i_p\}$ is fixed for the pair $\{U,\sigma\}$. We construct the inclusion $\theta(U,\sigma) : \widetilde{G^p}(U,\sigma) \to G^p(U,\sigma)$. To this end denote

for a given substitution $\omega \in S_{p+1}$ by means of ρ_ω, the canonical isomorphism

$$\rho_\omega : \widetilde{G}_{[i_0,\dots,i_p]} \to \widetilde{G}_{[i_{\omega(0)},\dots,i_{\omega(p)}]}.$$

Put

$$\theta(U,\sigma)\mid_{\widetilde{G}_{[i_0,\dots,i_p]}} = \prod_{\omega \in S_{p+1}} (\rho_\omega \times (-1)^{\operatorname{sign}\omega}).$$

It is not hard to verify that

$$\operatorname{Im}(\gamma^{U,\sigma}_{(U',\sigma')} \circ \theta(U,\sigma)) \subseteq \operatorname{Im}\theta(U',\sigma')$$

and, consequently, $\{\widetilde{G^p}(U,\sigma)\}$ may be regarded as a partial inductive system of sheaves. The limit of this system $\widetilde{G^p}$ is the desired subsheaf of the sheaf G^p.

Lemma 4. (G^p) and $(\widetilde{G^p})$ are resolutions of the sheaf L, and the sheaf $G^0 = \widetilde{G}^0 = f_* L$.

Hence, in a known way, one obtains

Theorem 1. If $f : X \to Y$ is a closed finite-to-one mapping of a paracompact space X onto a paracompact space Y, then for every open $U \subseteq Y$ there exists a spectral sequence converging to the suitably filtered group $H^*(Y; L_U)$, whose term $E_2^{p,q}$ is $H^p(H^q(Y; \widetilde{G}^*_U))$, where $\widetilde{G}^p$ are the sheaves described above.

For a finite-to-one mapping $f : X \to Y$ let us agree to denote

$$X_p^+ = \{x \mid f^{-1}fx \text{ contains } \leq p \text{ points}\}, \quad X_p^- = X \setminus X_{p+1}^+, \quad X_p^0 = X_p^+ \cap X_p^-.$$

Lemma 5.

$$\dim_{G^p} Y \leq r\ \dim_L X_{p+1}^+.$$

Here

$$r \dim_L M = \sup_{F \subseteq M} \{\dim_L F\}$$

(F closed in X) is the relative dimension of the set $M \subseteq X$.

The lemma follows from the fact that the direct image $f_* L$ of the constant sheaf L is the limit of a regular partial inductive system of sheaves of the form

$$\bigoplus_i L_{F_i} \mid U$$

(F_i closed in U), and from the fact that the tensor product of a sheaf whose stalks are free L-modules with a sheaf of this form preserves dimension, in view of Lemma 2.

From Theorem 1 and Lemma 5 one obtains in the usual way

Theorem 2. If there exists a closed mapping of multiplicity $\leq k + 1$ of a paracompact space X onto a paracompact space Y, then

$$\dim_L Y \leq \max_{0 \leq p \leq k} \{r \dim_L X_{p+1}^+ + p\}.$$

Corollary 1. Under the assumptions of Theorem 2,

$$\dim Y \leq \max_{0 \leq p \leq k} \{r \dim X_{p+1}^+ + p\}.$$

Corollary 2. Under the assumptions of Theorem 2, if the multiplicity of the closed mapping $f \leq k + 1$, then

$$\dim Y \leq \dim X + k.$$

Let us note that if $f : X \to Y$ is a closed mapping of multiplicity $\leq k + 1$ of a normal space X onto a normal space Y, then the associated mapping $\beta(f) : \beta X \to \beta Y$ of the Čech extensions is also a closed mapping of multiplicity $\leq k + 1$. This follows easily, for example, from the description of the Wallman compactification and its coincidence with the Čech compactification for normal spaces. Hence the following theorem is derived also from Corollary 2.

Theorem 3. If there exists a closed mapping of a normal space X onto a normal space Y of multiplicity $\leq k+1$, then

$$\dim X \leq \dim Y + k.$$

Two theorems for cohomological manifolds. The following is based on Lemma 6, its corollaries, and known properties of cohomological manifolds (^{9}).

We shall say that a mapping f of a space X is k-simple if it is finite-to-one and all the sets X_s^0 are empty, except for X_k^0 and, possibly, X_1^0.

Lemma 6. If f is a k-simple open-closed* mapping of a space X onto a space Y, then for every locally closed $A \subseteq Y$ there is defined a homomorphism

$$\sigma : f_* L_{f^{-1}A} \to L_A$$

such that: 1) for any open sets H and G, $H \subseteq G \subseteq Y$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & f_* L_{f^{-1}H} \arrow[r] \arrow[d, "\sigma"] & f_* L_{f^{-1}G} \arrow[r] \arrow[d, "\sigma"] & f_* L_{f^{-1}(G \setminus H)} \arrow[r] \arrow[d, "\sigma"] & 0, \\
0 \arrow[r] & L_H \arrow[r] & L_G \arrow[r] & L_{(G \setminus H)} \arrow[r] & 0;
\end{tikzcd}$$

2) the homomorphism σ on X_1^0 is multiplication in the sheaf $f_* L_{X_1^0} = L_{fX_1^0}$ by k;

3) the sequence $f_* L_{f^{-1}A}$ is exact if $A \subseteq f X_1^0$.

Theorem 4. The set of points of maximal multiplicity of an open-closed mapping of bounded multiplicity of a connected cohomological manifold X is everywhere dense in X.

Theorem 5. A finite-multiplicity open-closed mapping of a connected cohomological manifold has bounded multiplicity.

The plan of proof of these theorems is approximately the same as in (^{8}), but the details are substantially different.

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* A mapping is open-closed if it is open and closed simultaneously. A mapping is open if the image of every open set is open.

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Note: Figure translations are in progress. See original paper for figures.