ON SPECTRAL TRAJECTORIES GENERATED BY UNITARY EXTENSIONS OF ISOMETRIC SHIFT OPERATORS IN A FINITE-DIMENSIONAL SPACE \(\Pi_1\)

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ON SPECTRAL TRAJECTORIES GENERATED BY UNITARY EXTENSIONS OF ISOMETRIC SHIFT OPERATORS IN A FINITE-DIMENSIONAL SPACE Π_1

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The present note is directly adjacent to our preceding publication (^{1}), detailing the case which, in terms of sequences $\{c_p\}_0^{n-1}$ of the class $\mathfrak{P}_{1;n}$, was there called the case of **continuation** of rank n. Here, however, we shall use predominantly operator-theoretic terminology, while retaining all the definitions and notation introduced in (^{1}).

1. Let Π_χ be a Pontryagin space of finite dimension n ($\geqslant 2$), in which a certain basis $\{e_p\}_0^{n-1}$ is given, and let V be the isometric (with respect to the indefinite metric (x, y) of the space Π_χ) shift operator (of this basis):

$$Ve_p = e_{p+1} \quad (p = 0, 1, \dots, n-2),$$

   with defect index $(1, 1)$.* Under these conditions the Gram matrix $\|(e_p, e_q)\|_{p,q=0}^{n-1}$ is Toeplitz, i.e. has the form

$$\|c_{p-q}\|_{p,q=0}^{n-1} \quad (c_{p-q} = (e_p, e_q),\ p, q = 0, 1, \dots, n-1),$$

   and the corresponding Toeplitz form

$$\sum_{p,q=0}^{n-1} c_{p-q}\xi_p\bar{\xi}_q$$

   is nondegenerate ($\Delta_{n-1} \neq 0$) and has χ positive squares. Here we have used the notation

$$\Delta_k = \det \|c_{p-q}\|_{p,q=0}^{k} \quad (k = 0, 1, \dots, n-1).$$

   It is easy to see that, conversely, a Toeplitz form with the described properties determines both the space Π_χ and the isometric shift operator V acting in it (^{1},^{2}).

The operator V admits an infinite set of unitary extensions "with exit" and without exit from the given space Π_χ (^{1}); of these, only the latter are considered here. These extensions, for $\chi = 1$, may be of three types: elliptic, parabolic, and hyperbolic, and each of these three cases is characterized by certain spectral properties of the unitary operator $U(\supset V)$ (^{1}).

2. The effective construction in the space Π_χ of all unitary extensions of the operator V can be carried out as follows. Form the equation

$$(\Delta_n(\xi) \equiv) \begin{vmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_{n-1} & \bar{\xi} \\ c_1 & c_0 & \cdots & \bar{c}_{n-2} & \bar{c}_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_{n-2} & \cdots & c_0 & \bar{c}_1 \\ \xi & c_{n-1} & \cdots & c_1 & c_0 \end{vmatrix} = 0, \tag{1}$$

which defines in the ξ-plane, for $\Delta_{n-2} \neq 0$, a circle of positive radius

$$\rho = |\Delta_{n-1}| \cdot |\Delta_{n-2}|^{-1},$$

and for $\Delta_{n-2} = 0$, a straight line (^{2}).

* It is easy to see that, in order that an arbitrary linear operator V with defect index $(1,1)$, given in a finite-dimensional space, could be realized as the shift operator of a basis of this space, it is necessary and sufficient that V have no eigenvectors.

Theorem 1. The set of all unitary extensions of the operator V in the space Π_χ coincides with the set of operators $U = U_\xi$, each of which is given in the basis $\{e_p\}_0^{n-1}$ by the matrix

$$U_\xi = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ -\dfrac{A_n(\xi)}{A_0} & -\dfrac{A_{n-1}(\xi)}{A_0} & \cdots & -\dfrac{A_1(\xi)}{A_0} \end{pmatrix}, \tag{2}$$

where $(\Delta_{n-1}) = A_0,\ A_1(\xi), \dots, A_{n-1}(\xi),\ A_n(\xi)$ are the algebraic complements of the elements $c_0, \bar{c}_1, \dots, \bar{c}_{n-1}, \bar{\xi}$ of the first row of the determinant $\Delta_n(\xi)$, respectively, and ξ is an arbitrary root of equation (1).

Corollary. The characteristic equation of the operator U_ξ has the form

$$\begin{vmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_{n-1} & 1 \\ c_1 & c_0 & \cdots & \bar{c}_{n-2} & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_{n-2} & \cdots & c_0 & \lambda^{n-1} \\ \xi & c_{n-1} & \cdots & c_1 & \lambda^n \end{vmatrix} = A_0\lambda^n + A_1(\xi)\lambda^{n-1} + \cdots + A_{n-1}(\xi)\lambda + A_n(\xi) = 0. \tag{3}$$

The assertion of Theorem 1 on the unitarity of the operator U_ξ, under condition (1), is a special case of a more general fact; namely, the following holds.

Theorem 2. The operator U_ξ (an extension of the operator V), given in the basis $\{e_p\}_0^{n-1}$ by the matrix (2), for arbitrary complex ξ, has the property that for any vector $x = \xi_0 e_0 + \xi_1 e_1 + \cdots + \xi_{n-2} e_{n-2} + \xi_{n-1} e_{n-1} \in \Pi_\chi$ the equality

$$(U_\xi x, U_\xi x) = (x, x) - |\xi_{n-1}|^2 \frac{\Delta_n(\xi)}{\Delta_{n-1}}. \tag{4}$$

holds.

It follows from Theorem 2 that in each of the two domains into which the ξ-plane is divided by the curve (or straight line) (1), the operator-function U_ξ is, with respect to the metric (x, y), respectively nonexpanding or noncontracting, and on the line (1) itself it is a unitary operator*. Moreover, if $\Delta_n(\xi) \neq 0$, then vectors $x \in \Pi_\chi$ with coordinate $\xi_{n-1} \neq 0$ are (strictly) stretched or compressed by the operator U_ξ in the corresponding domains. This fact plays an important role in establishing the formulation of Theorem 3 below.

3. Returning to the operator U_ξ for ξ satisfying equation (1), note that the roots of the characteristic equation (3) are situated symmetrically with respect to the unit circle; moreover, for $\chi = 1$ no more than one pair of them lies off this circle. The operator U_ξ has simple spectrum. Therefore the elliptic case is completely characterized by the absence of multiple roots of equation (3) and of roots not lying on the unit circle $|\lambda| = 1$; the parabolic case by the presence of a multiple root λ with $|\lambda| = 1$; and the hyperbolic case by a pair of roots $\{\lambda, \bar{\lambda}^{-1}\}$, $0 < |\lambda| \neq 1$. A root λ of equation 3 with $|\lambda| = 1$ will be called a root of the first (second) kind if the corresponding eigenvector $f = \lambda^{-1} U_\xi f$ of the operator U_ξ is positive (negative) (cf. (4)). It is clear that for $\chi = 1$ there exists at most one root of the first kind (the elliptic case is characterized by its presence).

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* It is not difficult to verify that the operator U_ξ introduced by us (for arbitrary ξ) essentially coincides with the operator T considered in the work of M. G. Krein (^{3}), while formula (4) essentially coincides with formula (17) of the same work, where it is obtained from the weightier general considerations and is then applied to the derivation of Theorems 1-3 of that work. We note that the proof of these theorems is simplified if, instead of the operator T, one immediately considers the operator U_ξ given by the matrix (2).

As was shown in (4), the roots of equation (3) are analytic functions of the parameter ξ, single-valued on a certain collection of finite-sheeted Riemann surfaces. From Theorem 2 and a slight modification of Theorem 4.2 of the paper by M. G. Krein (4) it follows that

Theorem 3. *When the point ξ moves continuously in one of the directions along the line (1), the roots of the first kind of equation (3), if such roots exist, and the roots of the second kind (which, for $n > 2\chi + 1$, necessarily exist) move continuously along the circle $|\lambda| = 1$ in opposite (and type-dependent) directions.*

4. In (1) we established (Theorem 7) that the operator V always has hyperbolic extensions U_ξ, i.e., for some values of the parameter ξ there is necessarily a pair of roots λ and $\bar{\lambda}^{-1}$ of equation (3) that has "jumped off" the circle $|\lambda| = 1$. They, as is easy to see, also vary continuously as ξ varies and, when ξ moves along the line (1), describe in the λ-plane certain curves with two branches, each mirror-symmetric with respect to the circle $|\lambda| = 1$ (an inner and an outer branch). We shall call these curves **spectral trajectories**. From Theorem 7 of (1) it follows that the equation of the spectral trajectories in polar coordinates has the form

$$\operatorname{Det} \left\| c_{p-q+1} e^{-i\varphi} - (r + 1/r) c_{p-q} + c_{p-q-1} e^{i\varphi} \right\|_{p,q=0}^{n-2} = 0.$$

Theorem 4. *If $\Delta_{n-2} \neq 0$, then all spectral trajectories are bounded; and if $\Delta_{n-2} = 0$, then there always exists a spectral trajectory for the outer branch of which some straight line is an asymptote (in both directions), and, consequently, for the inner branch the origin is an asymptotic point.*

The equation of the indicated asymptote in polar coordinates has the form $Ae^{-i\varphi} + \bar{A}e^{i\varphi} = 0$, where A is the minor of the element $\bar{c}_1$ in the first row of the determinant Δ_{n-1}. The fact that $A \neq 0$ when $\Delta_{n-2} = 0$ is proved with the help of a corollary of the general rule for counting the signatures of Toeplitz forms, established in (5), namely:

Theorem 5. *If the Toeplitz form*

$$\sum_{p,q=0}^{n-1} c_{p-q} \xi_p \bar{\xi}_q$$

is nondegenerate ($\Delta_{n-1} \neq 0$), π is the number of its positive squares and ν the number of its negative squares, and $\chi = \min\{\pi, \nu\}$, then in the sequence of numbers

$$1, \Delta_1, \Delta_2, \dots, \Delta_{n-2}, \Delta_{n-1} \tag{5}$$

there cannot be more than $2\chi - 1$ zeros. If there are exactly $2\chi - 1$ such zeros, then, after deleting from the sequence (5) all zeros and all isolated nonzero numbers, the signs within each of the remaining groups of numbers strictly alternate when $\chi = \pi$, and coincide when $\chi = \nu$.

Corollary. *For $\chi = \pi = 1$, the sequence (5) contains no more than one zero. If it is present, the signs of all the other members of this sequence strictly alternate.*

5. As is clear from the examples, cases are possible (cf. (1)) in which, for all ξ satisfying equation (1), the operators U_ξ are of hyperbolic type. In this

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

case the spectral trajectories are situated strictly inside and outside the circle $|\lambda| = 1$ (they have no common points with it). In other cases they may approach it (be tangent to it) and intersect it. If $\Delta_{n-2} \neq 0$, then each of the branches of a spectral trajectory that does not intersect the unit circle (in the case when there are no elliptic extensions U_ξ) is closed. Such, for example, for $n = 3$ and $c_0 = -1,\ c_1 = 0,\ c_2 = 4$, are the spectral trajectories $(r+1/r)^2 = 17-8\cos 2\varphi$ (Fig. 1a); for $c_0 = 0,\ c_1 = 1,\ c_2 = 5$, the spectral trajectories are $r + 1/r = 5\cos\varphi\pm$

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* We have preferred this illustrative formulation to a stricter one, which is not difficult to state following the

$\pm\sqrt{29\cos^2\varphi - 25}$, and for $c_0 = 0,\ c_1 = 1,\ c_2 = 4$, the spectral trajectories $r + 1/r = 4\cos\varphi \pm \sqrt{20\cos^2\varphi - 16}$, shown schematically in Figs. 1 and 1 , respectively. In Fig. 1 , the point of tangency of the outer and inner branches corresponds to the operator U_ξ of parabolic type.

Altogether, for a given operator U, parabolic extensions can correspond only to a finite number ($\leq 2(n-1)$) of values of the parameter ξ ((1), Theorem 6). This number is certainly positive if elliptic extensions U_ξ correspond to certain ξ' s ((1), Theorem 8). As follows from Theorem 3, parabolic extensions U_ξ correspond in every case to those values of the parameter ξ for which, on the circle $|\lambda| = 1$, roots of the first and second kind of equation (3), moving in opposite directions, collide.

Fig. 1

For example, for $c_0 = -1,\ c_1 = 0,\ c_2 = 1+i$, the spectral trajectories $(r+1/r)^2 = 3 - 2\cos 2\varphi + 2\sin 2\varphi$ meet the circle $|\lambda| = 1$

Fig. 2

at four points determined by the equation $\sin(2\varphi - \pi/4) = \sqrt{2}/4$, and giving four critical numbers (1) for the four parabolic extensions U_ξ of the operator V, respectively.

As for the case when $\Delta_{n-2} = 0$, for $n = 3$ it is illustrated schematically in Fig. 2, which corresponds to simple numerical examples.

Of course, for $n > 3$ the picture of spectral trajectories, characterized in general by Theorem 4, will be more varied.

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Note: Figure translations are in progress. See original paper for figures.

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