Recurrent solutions of differential equations and the general theory of dynamical systems

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Abstract

The results published previously (Doklady Akademii Nauk SSSR, 167, No. 5, pp. 1004–1007) are presented in detail. Bibliography: 20 items.

Full Text

Preamble

This work, following the foundational methods established in [1–4] and [7], addresses the qualitative analysis of differential equations in functional spaces. Building upon the results of [5, 6], we examine the properties of solutions within the framework of topological structures. Specifically, we investigate the behavior of solutions to equation (13) as discussed in [8] and [9]. The stability and asymptotic behavior of these solutions are analyzed under various conditions, particularly focusing on the properties of the shift operator and the compactness of trajectories in the function space $\mathcal{F}$, as defined in [15].

We consider a mapping f from T into the space $(X;Y)$, where $t \to f_t$ represents a continuous dependence as outlined in [16]. Let X be a compact space and Y a metric space. We define the space of continuous mappings $(T;Y)$ and $(X;Y)$ with the topology of uniform convergence. Following the definitions in [15], the distance between two functions ϕ and ψ is given by:

$$\rho(\phi, \psi) = \sup \min\{\rho(\phi(t), \psi(t)), 1\}$$

This metric ensures that the space $(T;Y)$ is complete if Y is complete.

2.1 Stability and Convergence

In the context of the work by M. V. Belyaev [5, 6], we consider the conditions under which a solution ϕ is stable. Specifically, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\rho(\phi, \psi) < \delta$, then the supremum of the distance between the trajectories remains bounded by ϵ for all t. This leads to the definition of almost-periodic functions in the sense of Bohr and Levitan, as discussed in [17].

We define the operator $G(t, x)$ acting on the domain $D \subset E$. The convergence of the sequence of functions $\{h_n\}$ to f in the space $(T; (D; B))$ implies that the corresponding solutions ϕ_n of the differential equations:

$$x' = h_n(t)x$$

converge to the solution of the limit equation $x' = f(t)x$. As demonstrated in [15, p. 46], if f and g are continuous mappings, the distance between their images under the shift operator satisfies:

$$\rho(f^*, g^*) = \sup_{|t|<l} \rho(f_t, g_t)$$

This relationship is critical for establishing the existence of almost-periodic solutions for the system (13).

2.2 Asymptotic Properties

Consider the linear system $x' = A(t)x + a(t)$. According to the results in [19] and [20], if the coefficient matrix $A(t)$ and the vector function $a(t)$ satisfy certain regularity conditions, the existence of a bounded solution on the interval $[t_0, +\infty)$ implies the existence of an almost-periodic solution. This is further supported by the fixed-point theorems applied to the operator T in the space of continuous functions.

The analysis of the equation $x' = (x-1)x$ for $0 < x < 1$ serves as a representative example of the behavior described in Theorem 3. While the conditions for stability are met, the global behavior of the trajectory depends heavily on the initial values and the compactness of the set $D = \phi_0(J)$.

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Figures

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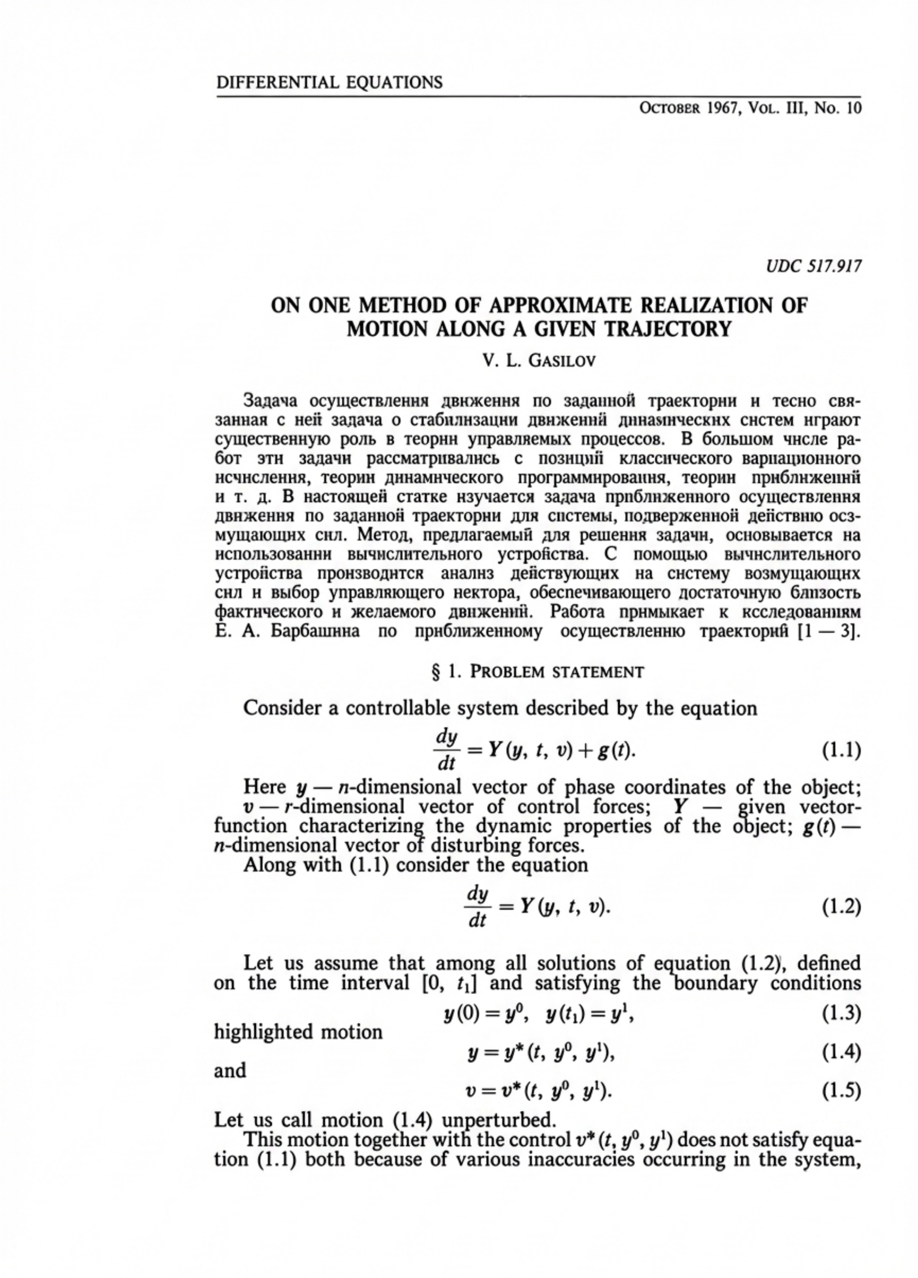

DIFFERENTIAL EQUATIONS

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ON ONE METHOD OF APPROXIMATE REALIZATION OF MOTION ALONG A GIVEN TRAJECTORY

V. L. Gasilov

Задача осуществления движения по заданной траектории и тесно связанная с ней задача о стабилизации движений динамических систем играют существенную роль в теории управляемых процессов. В большом числе работ эти задачи рассматривались с позиций классического вариационного исчисления, теории динамического программирования, теории приближений и т. д. В настоящей статке изучается задача приближенного осуществления движения по заданной траектории для системы, подверженной действию осзмущающих сил. Метод, предлагаемый для решения задачи, основывается на использовании вычислительного устройства. С помощью вычислительного устройства производится анализ действующих на систему возмущающих сил и выбор управляющего нектора, обеспечивающего достаточную близость фактического и желаемого движений. Работа примыкает к ксследованиям Е. А. Барбашина по приближенному осуществлению траекторий [1 — 3].

§ 1. Problem statement

Consider a controllable system described by the equation

$$\frac{dy}{dt} = \boldsymbol{Y}(\boldsymbol{y}, t, \boldsymbol{v}) + \boldsymbol{g}(t). \tag{1.1}$$

Here $\boldsymbol{y}$ — n-dimensional vector of phase coordinates of the object; $\boldsymbol{v}$ — r-dimensional vector of control forces; $\boldsymbol{Y}$ — given vector-function characterizing the dynamic properties of the object; $\boldsymbol{g}(t)$ — n-dimensional vector of disturbing forces.

Along with (1.1) consider the equation

$$\frac{dy}{dt} = \boldsymbol{Y}(\boldsymbol{y}, t, \boldsymbol{v}). \tag{1.2}$$

Let us assume that among all solutions of equation (1.2), defined on the time interval $[0, t_1]$ and satisfying the boundary conditions

$$\boldsymbol{y}(0) = \boldsymbol{y}^0, \quad \boldsymbol{y}(t_1) = \boldsymbol{y}^1, \tag{1.3}$$

highlighted motion

$$\boldsymbol{y} = \boldsymbol{y}^*(t, \boldsymbol{y}^0, \boldsymbol{y}^1), \tag{1.4}$$

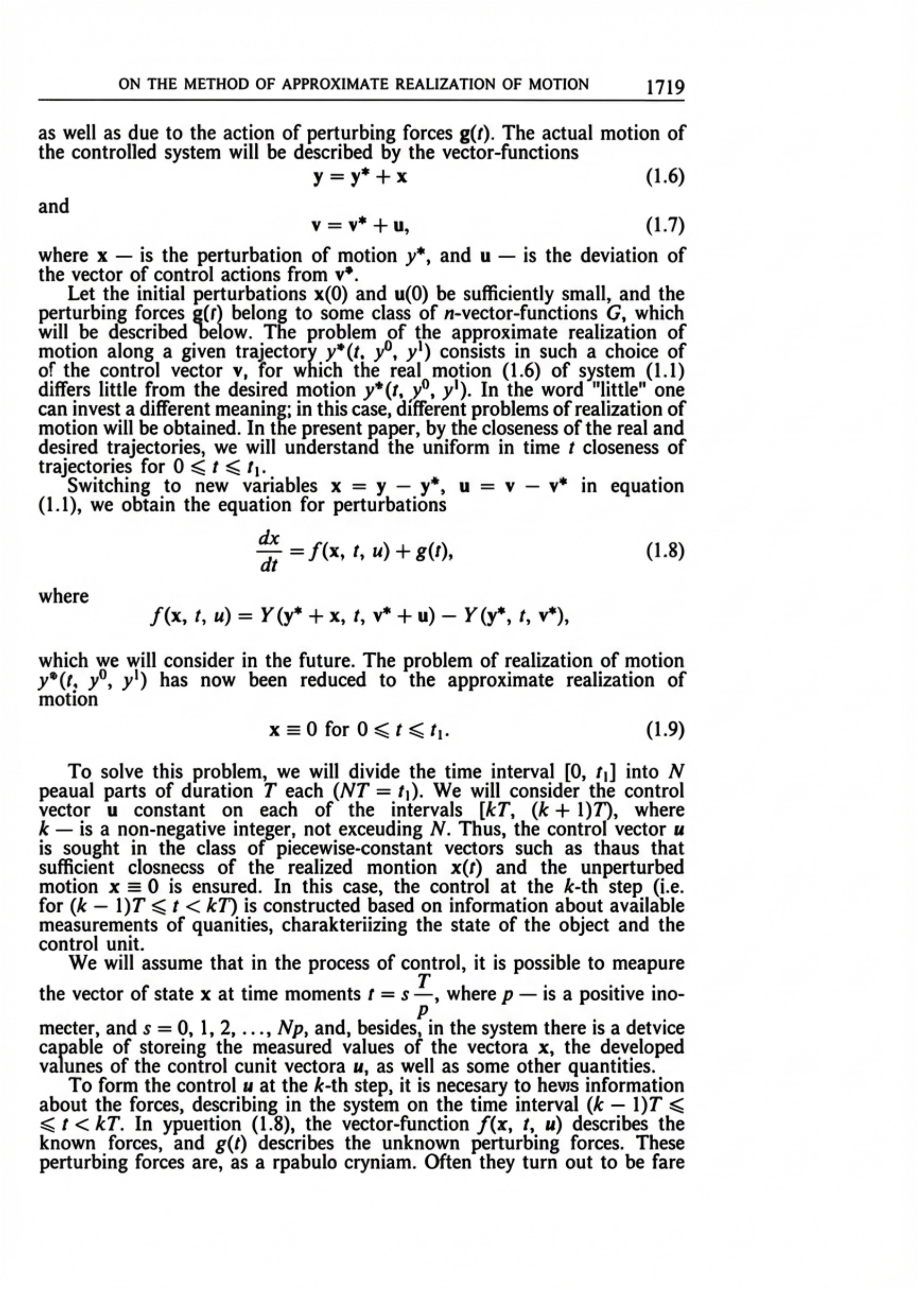
and

$$\boldsymbol{v} = \boldsymbol{v}^*(t, \boldsymbol{y}^0, \boldsymbol{y}^1). \tag{1.5}$$

Let us call motion (1.4) unperturbed.

This motion together with the control $\boldsymbol{v}^*(t, \boldsymbol{y}^0, \boldsymbol{y}^1)$ does not satisfy equation (1.1) both because of various inaccuracies occurring in the system,

Figure 1: Figure 1



as well as due to the action of perturbing forces $\mathbf{g}(t)$. The actual motion of the controlled system will be described by the vector-functions

$$\mathbf{y} = \mathbf{y}^* + \mathbf{x} \tag{1.6}$$

and

$$\mathbf{v} = \mathbf{v}^* + \mathbf{u}, \tag{1.7}$$

where $\mathbf{x}$ — is the perturbation of motion y^*, and $\mathbf{u}$ — is the deviation of the vector of control actions from $\mathbf{v}^*$.

Let the initial perturbations $\mathbf{x}(0)$ and $\mathbf{u}(0)$ be sufficiently small, and the perturbing forces $\mathbf{g}(t)$ belong to some class of n-vector-functions G, which will be described below. The problem of the approximate realization of motion along a given trajectory $y^*(t, y^0, y^1)$ consists in such a choice of of the control vector $\mathbf{v}$, for which the real motion (1.6) of system (1.1) differs little from the desired motion $y^*(t, y^0, y^1)$. In the word "little" one can invest a different meaning; in this case, different problems of realization of motion will be obtained. In the present paper, by the closeness of the real and desired trajectories, we will understand the uniform in time t closeness of trajectories for $0 \leqslant t \leqslant t_1$.

Switching to new variables $\mathbf{x} = \mathbf{y} - \mathbf{y}^*$, $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ in equation (1.1), we obtain the equation for perturbations

$$\frac{dx}{dt} = f(\mathbf{x}, t, u) + g(t), \tag{1.8}$$

where

$$f(\mathbf{x}, t, u) = Y(\mathbf{y}^* + \mathbf{x}, t, \mathbf{v}^* + \mathbf{u}) - Y(\mathbf{y}^*, t, \mathbf{v}^*),$$

which we will consider in the future. The problem of realization of motion $y^*(t, y^0, y^1)$ has now been reduced to the approximate realization of motion

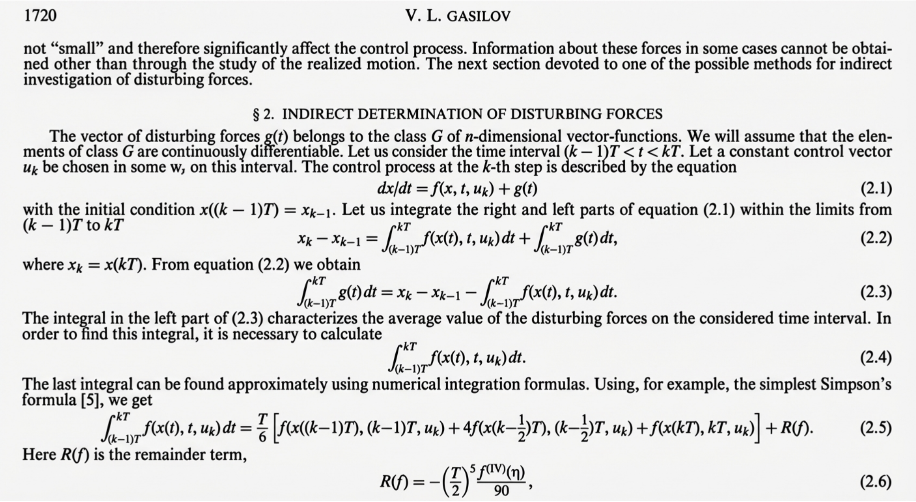
$$\mathbf{x} \equiv 0 \text{ for } 0 \leqslant t \leqslant t_1. \tag{1.9}$$

To solve this problem, we will divide the time interval $[0, t_1]$ into N peaual parts of duration T each ($NT = t_1$). We will consider the control vector $\mathbf{u}$ constant on each of the intervals $[kT, (k+1)T)$, where k — is a non-negative integer, not exceuding N. Thus, the control vector $\boldsymbol{u}$ is sought in the class of piecewise-constant vectors such as thaus that sufficient closnecss of the realized montion $\mathbf{x}(t)$ and the unperturbed motion $\mathbf{x} \equiv 0$ is ensured. In this case, the control at the k-th step (i.e. for $(k-1)T \leqslant t < kT$) is constructed based on information about available measurements of quanities, charakteriizing the state of the object and the control unit.

We will assume that in the process of control, it is possible to meapure the vector of state $\mathbf{x}$ at time moments $t = s\dfrac{T}{p}$, where p — is a positive inomecter, and $s = 0, 1, 2, \ldots, Np$, and, besides, in the system there is a detvice capable of storeing the measured values of the vectora $\mathbf{x}$, the developed valunes of the control cunit vectora $\boldsymbol{u}$, as well as some other quantities.

To form the control $\boldsymbol{u}$ at the k-th step, it is necesary to hevıs information about the forces, describing in the system on the time interval $(k-1)T \leqslant$ $\leqslant t < kT$. In ypuетion (1.8), the vector-function $f(\mathbf{x}, t, \boldsymbol{u})$ describes the known forces, and $g(t)$ describes the unknown perturbing forces. These perturbing forces are, as a rpabulo cryniam. Often they turn out to be fare

Figure 2: Figure 2



not “small” and therefore significantly affect the control process. Information about these forces in some cases cannot be obtained other than through the study of the realized motion. The next section devoted to one of the possible methods for indirect investigation of disturbing forces.

§ 2. INDIRECT DETERMINATION OF DISTURBING FORCES

The vector of disturbing forces $g(t)$ belongs to the class G of n-dimensional vector-functions. We will assume that the elements of class G are continuously differentiable. Let us consider the time interval $(k-1)T < t < kT$. Let a constant control vector u_k be chosen in some w, on this interval. The control process at the k-th step is described by the equation

$$dx/dt = f(x, t, u_k) + g(t) \tag{2.1}$$

with the initial condition $x((k-1)T) = x_{k-1}$. Let us integrate the right and left parts of equation (2.1) within the limits from $(k-1)T$ to kT

$$x_k - x_{k-1} = \int_{(k-1)T}^{kT} f(x(t), t, u_k)\,dt + \int_{(k-1)T}^{kT} g(t)\,dt, \tag{2.2}$$

where $x_k = x(kT)$. From equation (2.2) we obtain

$$\int_{(k-1)T}^{kT} g(t)\,dt = x_k - x_{k-1} - \int_{(k-1)T}^{kT} f(x(t), t, u_k)\,dt. \tag{2.3}$$

The integral in the left part of (2.3) characterizes the average value of the disturbing forces on the considered time interval. In order to find this integral, it is necessary to calculate

$$\int_{(k-1)T}^{kT} f(x(t), t, u_k)\,dt. \tag{2.4}$$

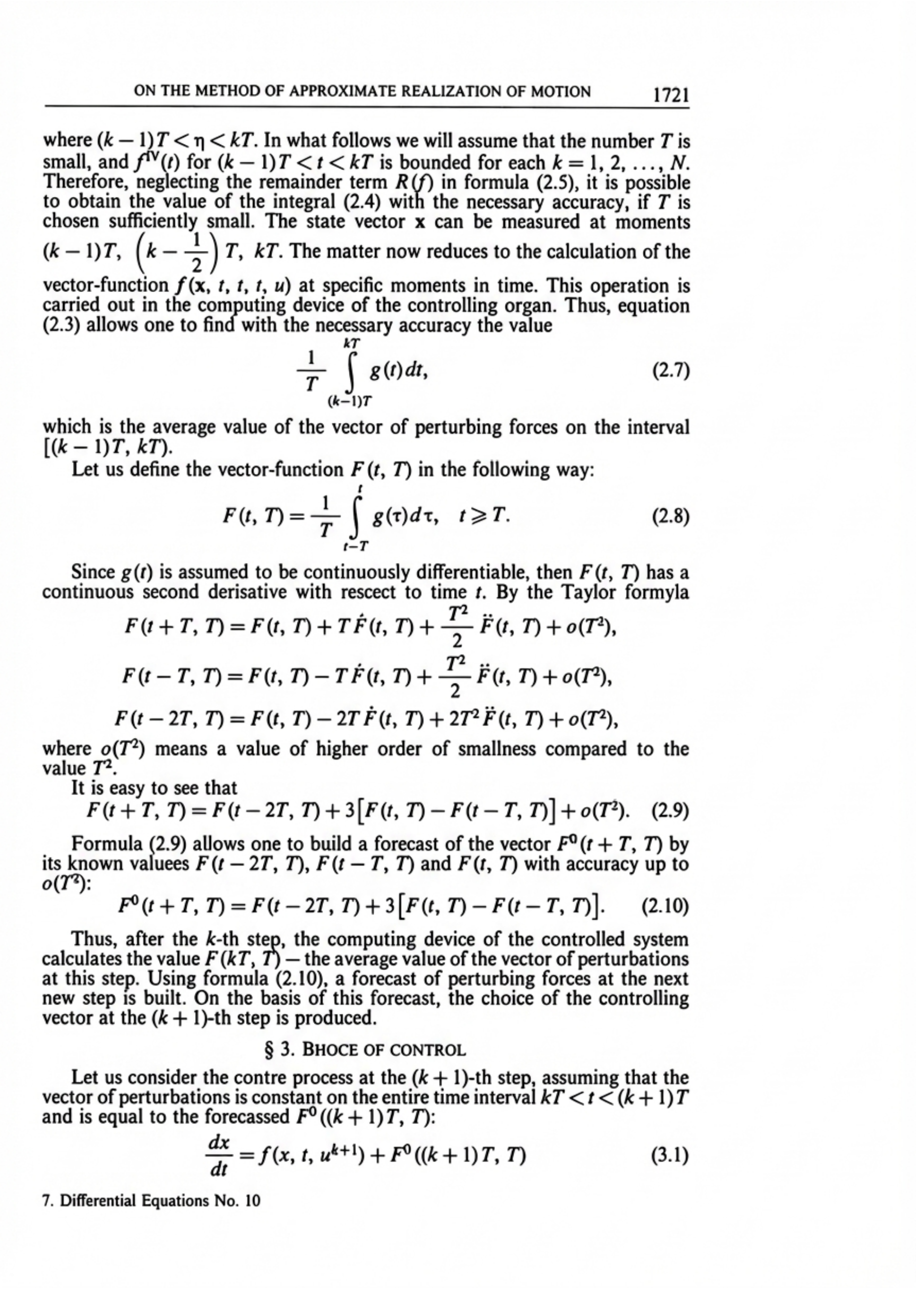
The last integral can be found approximately using numerical integration formulas. Using, for example, the simplest Simpson’s formula [5], we get

$$\int_{(k-1)T}^{kT} f(x(t), t, u_k)\,dt = \frac{T}{6}\left[f(x((k-1)T), (k-1)T, u_k) + 4f(x(k-\tfrac{1}{2})T), (k-\tfrac{1}{2})T, u_k) + f(x(kT), kT, u_k)\right] + R(f). \tag{2.5}$$

Here $R(f)$ is the remainder term,

$$R(f) = -\left(\frac{T}{2}\right)^5 \frac{f^{(\mathrm{IV})}(\eta)}{90}, \tag{2.6}$$

Figure 3: Figure 3



where $(k-1)T < \eta < kT$. In what follows we will assume that the number T is small, and $f^{\mathrm{IV}}(t)$ for $(k-1)T < t < kT$ is bounded for each $k = 1, 2, \ldots, N$. Therefore, neglecting the remainder term $R(f)$ in formula (2.5), it is possible to obtain the value of the integral (2.4) with the necessary accuracy, if T is chosen sufficiently small. The state vector $\mathbf{x}$ can be measured at moments $(k-1)T$, $\left(k - \frac{1}{2}\right)T$, kT. The matter now reduces to the calculation of the vector-function $f(\mathbf{x}, t, t, t, u)$ at specific moments in time. This operation is carried out in the computing device of the controlling organ. Thus, equation (2.3) allows one to find with the necessary accuracy the value

$$\frac{1}{T}\int_{(k-1)T}^{kT} g(t)\,dt, \tag{2.7}$$

which is the average value of the vector of perturbing forces on the interval $[(k-1)T, kT)$.

Let us define the vector-function $F(t, T)$ in the following way:

$$F(t, T) = \frac{1}{T}\int_{t-T}^{t} g(\tau)\,d\tau, \quad t \geqslant T. \tag{2.8}$$

Since $g(t)$ is assumed to be continuously differentiable, then $F(t, T)$ has a continuous second derisative with rescect to time t. By the Taylor formyla

$$F(t+T, T) = F(t, T) + T\dot{F}(t, T) + \frac{T^2}{2}\ddot{F}(t, T) + o(T^2),$$

$$F(t-T, T) = F(t, T) - T\dot{F}(t, T) + \frac{T^2}{2}\ddot{F}(t, T) + o(T^2),$$

$$F(t-2T, T) = F(t, T) - 2T\dot{F}(t, T) + 2T^2\ddot{F}(t, T) + o(T^2),$$

where $o(T^2)$ means a value of higher order of smallness compared to the value T^2.

It is easy to see that

$$F(t+T, T) = F(t-2T, T) + 3\left[F(t, T) - F(t-T, T)\right] + o(T^2). \tag{2.9}$$

Formula (2.9) allows one to build a forecast of the vector $F^0(t+T, T)$ by its known valuees $F(t-2T, T)$, $F(t-T, T)$ and $F(t, T)$ with accuracy up to $o(T^2)$:

$$F^0(t+T, T) = F(t-2T, T) + 3\left[F(t, T) - F(t-T, T)\right]. \tag{2.10}$$

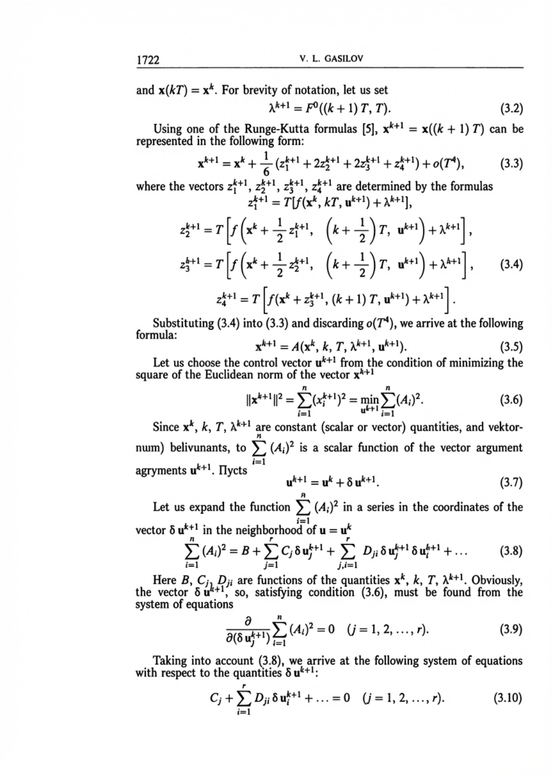
Thus, after the k-th step, the computing device of the controlled system calculates the value $F(kT, T)$ — the average value of the vector of perturbations at this step. Using formula (2.10), a forecast of perturbing forces at the next new step is built. On the basis of this forecast, the choice of the controlling vector at the $(k+1)$-th step is produced.

§ 3. BHOCE OF CONTROL

Let us consider the contre process at the $(k+1)$-th step, assuming that the vector of perturbations is constant on the entire time interval $kT < t < (k+1)T$ and is equal to the forecassed $F^0((k+1)T, T)$:

$$\frac{dx}{dt} = f(x, t, u^{k+1}) + F^0((k+1)T, T) \tag{3.1}$$

Figure 4: Figure 4



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and $\mathbf{x}(kT) = \mathbf{x}^k$. For brevity of notation, let us set

$$\lambda^{k+1} = F^0((k+1)\,T,\,T). \tag{3.2}$$

Using one of the Runge-Kutta formulas [5], $\mathbf{x}^{k+1} = \mathbf{x}((k+1)\,T)$ can be represented in the following form:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \frac{1}{6}(z_1^{k+1} + 2z_2^{k+1} + 2z_3^{k+1} + z_4^{k+1}) + o(T^4), \tag{3.3}$$

where the vectors z_1^{k+1}, z_2^{k+1}, z_3^{k+1}, z_4^{k+1} are determined by the formulas

$$\begin{gathered} z_1^{k+1} = T[f(\mathbf{x}^k,\,kT,\,\mathbf{u}^{k+1}) + \lambda^{k+1}], \\ z_2^{k+1} = T\left[f\left(\mathbf{x}^k + \frac{1}{2}z_1^{k+1},\ \left(k + \frac{1}{2}\right)T,\ \mathbf{u}^{k+1}\right) + \lambda^{k+1}\right], \\ z_3^{k+1} = T\left[f\left(\mathbf{x}^k + \frac{1}{2}z_2^{k+1},\ \left(k + \frac{1}{2}\right)T,\ \mathbf{u}^{k+1}\right) + \lambda^{k+1}\right], \\ z_4^{k+1} = T\left[f(\mathbf{x}^k + z_3^{k+1},\,(k+1)\,T,\,\mathbf{u}^{k+1}) + \lambda^{k+1}\right]. \end{gathered} \tag{3.4}$$

Substituting (3.4) into (3.3) and discarding $o(T^4)$, we arrive at the following formula:

$$\mathbf{x}^{k+1} = A(\mathbf{x}^k,\,k,\,T,\,\lambda^{k+1},\,\mathbf{u}^{k+1}). \tag{3.5}$$

Let us choose the control vector $\mathbf{u}^{k+1}$ from the condition of minimizing the square of the Euclidean norm of the vector $\mathbf{x}^{k+1}$

$$\|\mathbf{x}^{k+1}\|^2 = \sum_{i=1}^{n}(x_i^{k+1})^2 = \min_{\mathbf{u}^{k+1}}\sum_{i=1}^{n}(A_i)^2. \tag{3.6}$$

Since $\mathbf{x}^k$, k, T, λ^{k+1} are constant (scalar or vector) quantities, and vektornum) belivunants, to $\sum_{i=1}^{n}(A_i)^2$ is a scalar function of the vector argument agryments $\mathbf{u}^{k+1}$. Пycts

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\,\mathbf{u}^{k+1}. \tag{3.7}$$

Let us expand the function $\sum_{i=1}^{n}(A_i)^2$ in a series in the coordinates of the vector $\delta\,\mathbf{u}^{k+1}$ in the neighborhood of $\mathbf{u} = \mathbf{u}^k$

$$\sum_{i=1}^{n}(A_i)^2 = B + \sum_{j=1}^{r}C_j\,\delta\,\mathbf{u}_j^{k+1} + \sum_{j,i=1}^{r}D_{ji}\,\delta\,\mathbf{u}_j^{k+1}\,\delta\,\mathbf{u}_i^{k+1} + \ldots \tag{3.8}$$

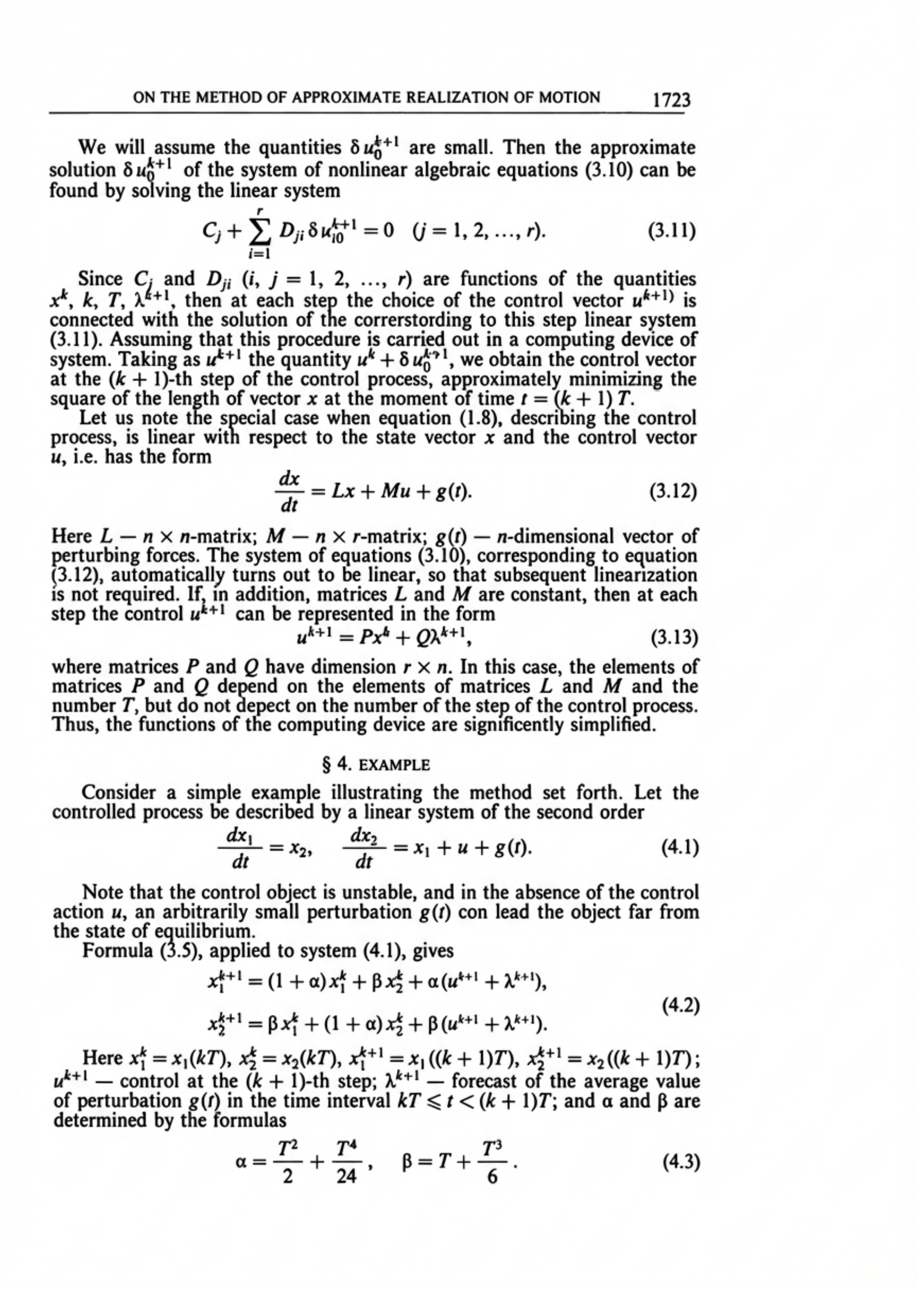
Here B, C_j, D_{ji} are functions of the quantities $\mathbf{x}^k$, k, T, λ^{k+1}. Obviously, the vector $\delta\,\mathbf{u}^{k+1}$, so, satisfying condition (3.6), must be found from the system of equations

$$\frac{\partial}{\partial(\delta\,\mathbf{u}_j^{k+1})}\sum_{i=1}^{n}(A_i)^2 = 0 \quad (j = 1, 2, \ldots, r). \tag{3.9}$$

Taking into account (3.8), we arrive at the following system of equations with respect to the quantities $\delta\,\mathbf{u}^{k+1}$:

$$C_j + \sum_{i=1}^{r}D_{ji}\,\delta\,\mathbf{u}_i^{k+1} + \ldots = 0 \quad (j = 1, 2, \ldots, r). \tag{3.10}$$

Figure 5: Figure 5



We will assume the quantities δu_0^{k+1} are small. Then the approximate solution δu_0^{k+1} of the system of nonlinear algebraic equations (3.10) can be found by solving the linear system

$$C_j + \sum_{i=1}^{r} D_{ji}\,\delta u_{i0}^{k+1} = 0 \quad (j = 1, 2, \ldots, r). \tag{3.11}$$

Since C_j and D_{ji} $(i,\ j = 1,\ 2,\ \ldots,\ r)$ are functions of the quantities x^k, k, T, λ^{k+1}, then at each step the choice of the control vector $u^{k+1)}$ is connected with the solution of the correrstording to this step linear system (3.11). Assuming that this procedure is carried out in a computing device of system. Taking as u^{k+1} the quantity $u^k + \delta u_0^{k+1}$, we obtain the control vector at the $(k + 1)$-th step of the control process, approximately minimizing the square of the length of vector x at the moment of time $t = (k + 1)\,T$.

Let us note the special case when equation (1.8), describing the control process, is linear with respect to the state vector x and the control vector u, i.e. has the form

$$\frac{dx}{dt} = Lx + Mu + g(t). \tag{3.12}$$

Here L — $n \times n$-matrix; M — $n \times r$-matrix; $g(t)$ — n-dimensional vector of perturbing forces. The system of equations (3.10), corresponding to equation (3.12), automatically turns out to be linear, so that subsequent linearization is not required. If, in addition, matrices L and M are constant, then at each step the control u^{k+1} can be represented in the form

$$u^{k+1} = Px^k + Q\lambda^{k+1}, \tag{3.13}$$

where matrices P and Q have dimension $r \times n$. In this case, the elements of matrices P and Q depend on the elements of matrices L and M and the number T, but do not depect on the number of the step of the control process. Thus, the functions of the computing device are significently simplified.

§ 4. EXAMPLE

Consider a simple example illustrating the method set forth. Let the controlled process be described by a linear system of the second order

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = x_1 + u + g(t). \tag{4.1}$$

Note that the control object is unstable, and in the absence of the control action u, an arbitrarily small perturbation $g(t)$ con lead the object far from the state of equilibrium.

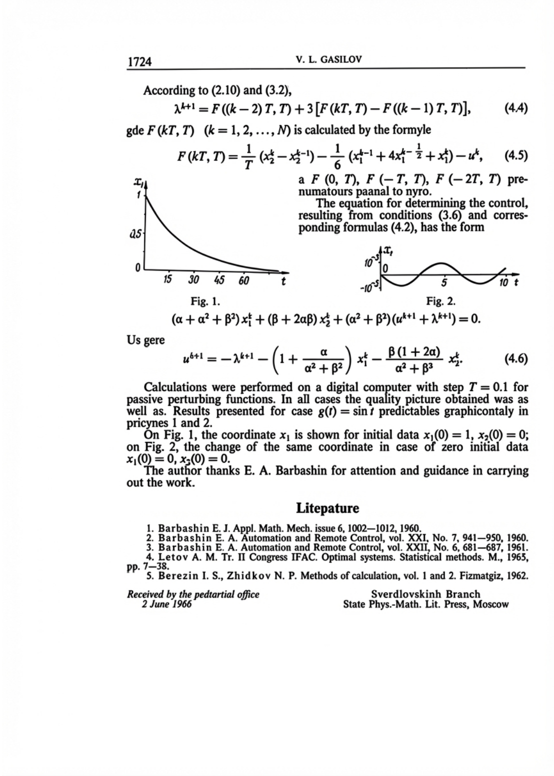
Formula (3.5), applied to system (4.1), gives

$$\begin{aligned} x_1^{k+1} &= (1 + \alpha)\,x_1^k + \beta\,x_2^k + \alpha\,(u^{k+1} + \lambda^{k+1}), \\ x_2^{k+1} &= \beta\,x_1^k + (1 + \alpha)\,x_2^k + \beta\,(u^{k+1} + \lambda^{k+1}). \end{aligned} \tag{4.2}$$

Here $x_1^k = x_1(kT)$, $x_2^k = x_2(kT)$, $x_1^{k+1} = x_1((k + 1)T)$, $x_2^{k+1} = x_2((k + 1)T)$; u^{k+1} — control at the $(k + 1)$-th step; λ^{k+1} — forecast of the average value of perturbation $g(t)$ in the time interval $kT \leqslant t < (k + 1)T$; and α and β are determined by the formulas

$$\alpha = \frac{T^2}{2} + \frac{T^4}{24}, \qquad \beta = T + \frac{T^3}{6}. \tag{4.3}$$

Figure 6: Figure 6



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According to (2.10) and (3.2),

$$\lambda^{k+1} = F((k-2)T, T) + 3\left[F(kT, T) - F((k-1)T, T)\right], \qquad (4.4)$$

gde $F(kT, T)$ $(k = 1, 2, \ldots, N)$ is calculated by the formyle

$$F(kT, T) = \frac{1}{T}\left(x_2^k - x_2^{k-1}\right) - \frac{1}{6}\left(x_1^{k-1} + 4x_1^{k-\frac{1}{2}} + x_1^k\right) - u^k, \qquad (4.5)$$

a $F(0, T)$, $F(-T, T)$, $F(-2T, T)$ prenumatours paanal to nyro.

The equation for determining the control, resulting from conditions (3.6) and corresponding formulas (4.2), has the form

Fig. 1.

Fig. 2.

$$(\alpha + \alpha^2 + \beta^2) x_1^k + (\beta + 2\alpha\beta) x_2^k + (\alpha^2 + \beta^2)(u^{k+1} + \lambda^{k+1}) = 0.$$

Us gere

$$u^{6+1} = -\lambda^{k+1} - \left(1 + \frac{\alpha}{\alpha^2 + \beta^2}\right) x_1^k - \frac{\beta(1 + 2\alpha)}{\alpha^2 + \beta^3} x_2^k. \qquad (4.6)$$

Calculations were performed on a digital computer with step $T = 0.1$ for passive perturbing functions. In all cases the quality picture obtained was as well as. Results presented for case $g(t) = \sin t$ predictables graphicontaly in pricynes 1 and 2.

On Fig. 1, the coordinate x_1 is shown for initial data $x_1(0) = 1$, $x_2(0) = 0$; on Fig. 2, the change of the same coordinate in case of zero initial data $x_1(0) = 0$, $x_2(0) = 0$.

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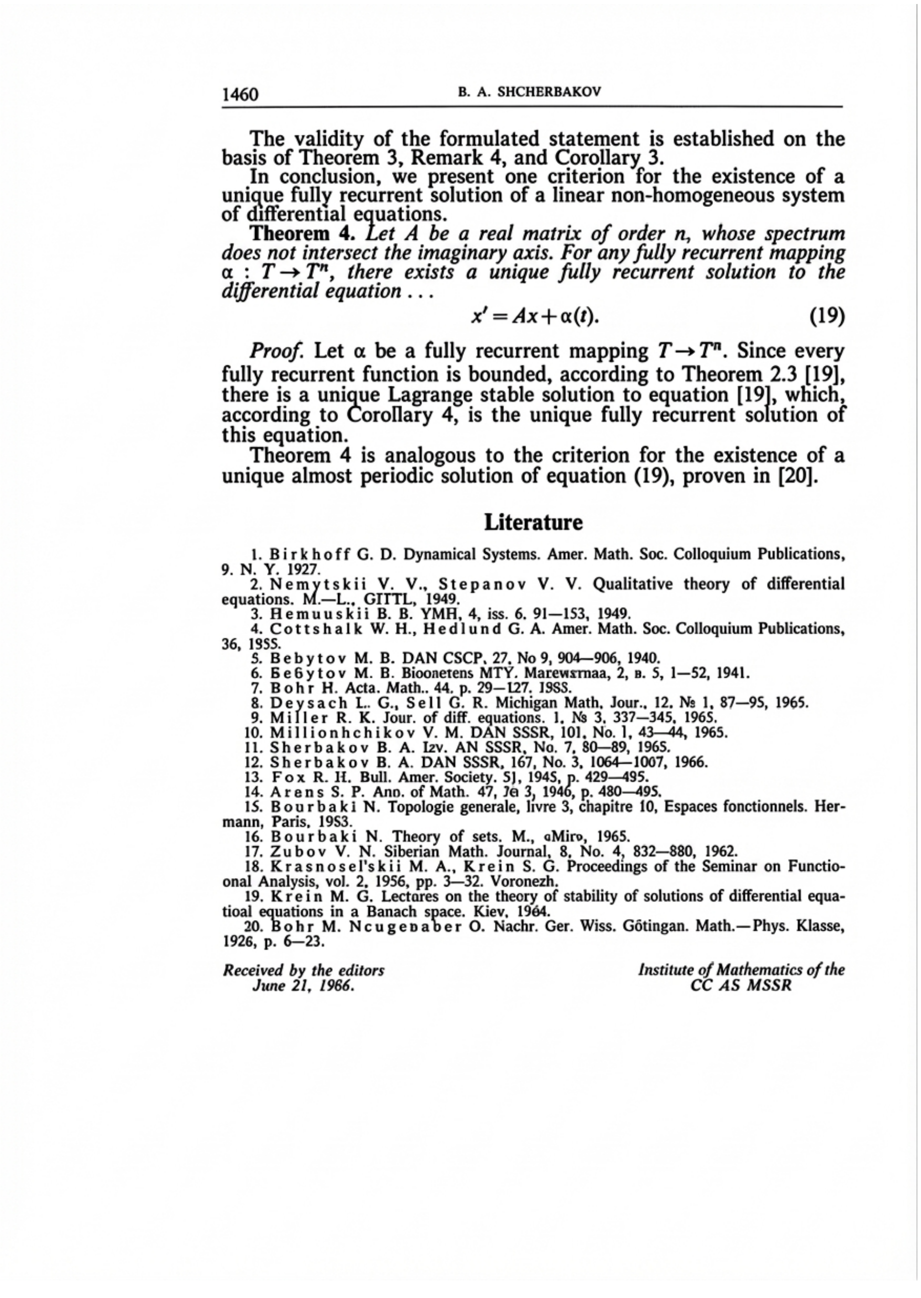
Figure 7: Figure 7



The validity of the formulated statement is established on the basis of Theorem 3, Remark 4, and Corollary 3.

In conclusion, we present one criterion for the existence of a unique fully recurrent solution of a linear non-homogeneous system of differential equations.

Theorem 4. *Let A be a real matrix of order n, whose spectrum does not intersect the imaginary axis. For any fully recurrent mapping* $\alpha : T \to T^n$, *there exists a unique fully recurrent solution to the differential equation . . .*

$$x' = Ax + \alpha(t). \tag{19}$$

Proof. Let α be a fully recurrent mapping $T \to T^n$. Since every fully recurrent function is bounded, according to Theorem 2.3 [19], there is a unique Lagrange stable solution to equation [19], which, according to Corollary 4, is the unique fully recurrent solution of this equation.

Theorem 4 is analogous to the criterion for the existence of a unique almost periodic solution of equation (19), proven in [20].

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Figure 8: Figure 8