Periodic generalized orbits

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Abstract

This work is related to the papers [1, 2, 3]. The problem of finding periodic generalized solutions for nonlinear generalized systems of differential equations of the type

$$\dot{\mu} = A(t)\mu + P(\mu, t) + \eta$$

is formulated and solved. Here $A(t)$ is a $n \times n$ matrix with infinitely differentiable elements, while η and μ represent the input and the response, respectively, belonging to the space K'_+ of generalized functions with supports located in the domain $t \geq 0$.

The "nonlinearity" on the right-hand side of system (1) is defined by a nonlinear operator P acting in the space K'_+. The input η has the following structure:

$$\eta = \mu_0 \delta^{(p)}(t) + \eta_1$$

Here μ_0 belongs to the Euclidean space E_n, $\delta^{(p)}(t)$ is the p-th derivative of the Dirac δ-function, and η_1 is a periodic generalized function having the integral representation

$$\langle \eta_1, x \rangle = \int_0^{ax^{(p)}} dn(t), \quad x \in K(a).$$

Bibliography: 8 items.

Full Text

Preamble

This work continues the investigations presented in [1, 2, 3]. Specifically, in [3], the concept of a v-stable solution was introduced. We consider the properties of such solutions and their relationship with various classes of functions. Following the methodologies established in [7, 8], we define the operator h acting on the space $K(a)$ as described in [4]. Let h be a parameter such that for $x \in K(a)$, the relation $x(t-h)$ satisfies the condition:

$$\langle \eta, x(t-h) \rangle \tag{1.1}$$

where η is a functional in the space K^*. We assume $h = 1$ and $t > 0$. Using the representation for the inner product:

$$\langle \eta, x \rangle = \int x(p^{\alpha}(t)) d\eta_{\alpha}(t) \quad (\alpha = 1, 2, \dots) \tag{1.2}$$

where η_α are components associated with the functional $\eta \in K^*$. For $x \in K(a)$ and $t > 0$, the shift property holds:

$$\langle \eta, x(t-1) \rangle = \langle \eta, x(t) \rangle$$

This implies that $\langle \eta, x(t-1) \rangle \in K(a+1)$. From (1.2), it follows that $x(t-1)$ can be represented as:

$$\langle \eta, x(t-1) \rangle = \int x(p^{\alpha+1}(t-1)) d\eta_{\alpha+1}(t)$$

By setting $\tau = t - 1$, we obtain the following relation for the functional $X(t)$:

$$\langle X(t), x \rangle = \int X(p^{\alpha+1})(t) d\eta_{\alpha+1}(t) \tag{1.3}$$

The evolution of the measure $\eta_{\alpha+1}$ is governed by:

$$\eta_{\alpha+1}(t+1) = \eta_{\alpha}(t) + Q_{p_{\alpha+1}}(t) \tag{1.4}$$

for $0 < t < a$ $(\alpha = 1, 2, \dots)$, where $Q_{p_{\alpha+1}}$ is a distribution on $K(a)$. Following the results in [2], we have:

$$\langle \eta, x \rangle = \int x^{(p_{\alpha+1})}(t) d[(-1)^{p_{\alpha+1}} \Delta^{p_{\alpha+1}} \eta_{\alpha+1}(t)]$$

The relationship between successive measures is given by:

$$\eta_{\alpha+1}(t) = (-1)^{\alpha} \eta_{\alpha}(t-\alpha) \tag{1.5}$$

where $\alpha = p_{\alpha+1} - p_\alpha \geq 2$. Thus, for $0 < t < a$, the measure satisfies:

$$\eta_{\alpha+1}(t) = \begin{cases} \eta_\alpha(t-1) + Q_{p_{\alpha+1}}(t-1), & a < t < a+1 \\ \dots \end{cases} \tag{1.6}$$

This construction ensures that for $1 < t < a$, the functional $\eta_{\alpha+1}(t)$ satisfies the required stability conditions.

2. Stability and Differential Equations

Consider the differential equation:

$$\dot{\mu} = A(t)\mu + P(\mu) + f(t) \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix and $P(\mu)$ is a nonlinear operator. We assume $P(\mu)$ satisfies a growth condition of the form:

$$\langle P(\mu), x \rangle = \int x^{(p)}(t) dR_p(\mu, t) \tag{2.2}$$

where R_p is a kernel representing the nonlinear part. Following the approach in [5] and [2], we define the transition operators:

$$G_{\alpha+1}(t) = G_{\alpha+1}(e^{\alpha} m_{\alpha+1}, p_{\alpha+1})(t) \tag{2.3}$$

where m_α are the corresponding moments. The operator R_α is defined as:

$$R_\alpha(m, p_\alpha)(t) = \begin{cases} (-1)^q p_\alpha \cdots, & 0 < t < a \\ G_{\alpha+1}(e^{\alpha} m, p_{\alpha+1})(t), & a < t < a+1 \end{cases} \tag{2.4}$$

For $\mu \in K^*$, the solution $m(t)$ can be expressed via the integral equation:

$$m(t) = -A_p(t,0)\mu_0 + \int_0^t A_p(t,s) R_p(m(s), s) ds - \int_0^t A_p(t,s) d\eta(s) \tag{2.7}$$

where $A_p(t,s)$ is the fundamental solution matrix satisfying:

$$A_p(t,s) = A_p(t+1, s+1) \tag{2.8}$$

and the exponential bound:

$$\|A_p(t,s)\| \leq A_0 e^{-\alpha(t-s)} \tag{2.9}$$

for $0 < A_0$ and $\alpha > 0$. We assume the nonlinearity R_p satisfies a Lipschitz condition:

$$\|R_p(m_1, t) - R_p(m_2, t)\|_E \leq L \|m_1 - m_2\|_E \tag{2.11}$$

Under these conditions, if the initial perturbation $\|\mu_0\|_E$ is sufficiently small, there exists a unique solution in the space N. Specifically, let $\chi = \alpha - LA_0 > 0$. Then the solution satisfies:

$$\|m(t)\|_E \leq A_0 e^{-\alpha t} \|\mu_0\|_E + A_0 L e^{-\alpha t} \int_0^t e^{\alpha s} \|m(s)\|_E ds + A_0 e^{-\alpha t} \int_0^t e^{\alpha s} d(\text{var } \eta) \tag{2.14}$$

Applying Gronwall’ s inequality, we obtain:

$$\|m(t)\|_E \leq (A_0 e^{-\lambda t} + \rho) e \tag{2.15}$$

where ρ depends on the variation of the noise term η. This estimate demonstrates the asymptotic stability of the solution. For any two initial conditions μ_{01} and μ_{02}, the corresponding solutions satisfy:

$$\|m(\mu_{01}, T) - m(\mu_{02}, T)\|_E \leq A_0 e^{-\lambda T} \|\mu_{01} - \mu_{02}\|_E \tag{2.17}$$

which implies the contractive property of the mapping for sufficiently large T.

3. Higher-Order Stability

If $p > q$, we consider the stability of the v-th order. Let $\Delta = \mu - \bar{\mu}$ be the deviation from the steady state. The equation for the deviation is:

$$\dot{\Delta} = A(t)\Delta + P(\Delta) + Y \tag{3.1}$$

where $P(\Delta) = P(\mu + \Delta) - P(\mu)$. The nonlinear term is represented as:

$$\langle P(\Delta), x \rangle = \int x^{(r-1)}(t) dR_r(f(s), s) \tag{3.3}$$

The kernel R_r is constructed using the moments of the lower-order terms. This formulation allows us to extend the stability results to the space $K^*(q)$, ensuring that the system remains stable under perturbations in higher-order derivatives.

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Figures

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DIFFERENTIAL EQUATIONS

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RECURRENT SOLUTIONS OF DIFFERENTIAL EQUATIONS AND THE GENERAL THEORY OF DYNAMICAL SYSTEMS

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In the present paper, we study such solutions of differential equations, called recurrent (see definition 1), which generate recurrent motions (see definition 2), and also generate recurrent functions in Birkhoff's sense [1—4] in a dynamical system of shifts of continuous functions.
The named dynamical system is a generalization of the universal dynamical system of the most sysnoes system of Bebutov [5,6] to the case when the phase space consists of continuous mappings of the real line into an arbitrary metric space.

By their properties, recurrent functions are more general than almost periodic ones (in the sense of Bohr [7]), and are clossly related to recurrent functions (see definition 1). So, for example, for a differential equation defining a dynamical system, the concepts of recurrent and fully recurrent solutions are analogous. The problem of studying solutions of differential equations generating recurrent motions in the Bebutov dynamical system was posed by V. V. Nemytskii ([3], p. 101).

One of the questions considered in this paper can be formulated as follows: under what conditions on its existence, the solution stable according to Lagrange in the positive direction of the differential equation (13) (see definition 5) implies the existence of fully recurrent solutions of this equation? An analogous problem for almost periodic solutions of a narrower class of differential equations was considered in [8] and [9].

If the right-hand side of equation (13) does not depend on the variable t and the conditions for the existence, uniqueness, and continuability of solutions on $(-\infty, +\infty)$ are satisfied, then the answer to the posed question follows from the well-known Birkhoff theorem, according to which in the ω-limit set of any solution stable according to Lagrange in the positive t-direction, there exists a minimal set of recurrent motions. In the general case, the answer to the posed question follows from the proven below theorem 2 (corollaries 2 and 3).

It is also clearly necessary to point out the proven in [9] sign of the existence of recurrent solutions of an autonomous differential equation in n-dimensional space. On the right-hand side of this equation are imposed conditions which actually ensure the existence not only of recurrent, but also of fully recurrent solutions.

Another question, considered in the present paper, can be formulated as follows: under what conditions does a solution of equation (13) become fully recurrent? The answer to this question is given by the proven here theorem 3 and corollary 4.

Based on

these propositions, a simple sign of the existence of a unique fully recurrent

Figure 1: Figure 1

of the solution of a linear nonhomogeneous system of differential equations (Theorem 4).

The issue of recurrent solutions of differential equations is devoted to works [10—12] and partially work [9]. The main results of this work are presented without proof in [12].

Below, as in [12], for the study of solutions of equations (13) as deposits to the dynamical system of shifts. This technique consists of the following: the solutions of this equation are considered as points of the phase space of a dynamical system in the space of continuous functions, whose law of motion is defined as shifts of functions. By applying methods of the general theory of dynamical systems, are studied motions, that are generated by the solutions of this equation. These motions fully characterize the properties of the considered solutions.

One technique for studying solutions of a differential equation proprovides an opportunity to use the results of the well-developed theory of dynamical systems at the present time, which leads to more general results with relatively simple proofs.

1°. Let us introduce the following notations: T — the real line; T_+ — the interval $[0, +\infty)$; T_- — the interval $(-\infty, 0]$; $\mathbf{N}$ — the set of all natural numbers; ρ — distance in metric spaces*).

Let X and Y be metric spaces.

By the symbol $(X; Y)$ we will denote the set of all continuous mappings of X into Y, endowed with the compact-open topology [13], which is also called the k-topology [14] and the topology of compact convergence [15].

Let φ be a continuous mapping of T into Y and $\tau \in T$. By the symbol φ^τ we denote the mapping $t \to \varphi(t+\tau)$ of the real line T into Y, by σ — the mapping $(\varphi, \tau) \to \varphi^\tau$ of the product $(T; Y) \times T$ into $(T; Y)$, and by the symbol $\sigma(T; Y)$ — the dynamical system in the space $(T; Y)$, defined by the mapping σ (see [4—6]).

Let f be a continuous mapping of $T \times X$ into Y. By the symbol f^* we will denote the mapping $t \to f_t$ of the real line T into $(X; Y)$, where f_t is the mapping defined by the function f for a fixed value t of the first argument (see [16]).

2°. Let us bring numerous definitions and a series of auxiliary proposits. Let us bring numerous definitions and a series of auxiliary proposition, related to them.

Let X be a compact metric space, Y be a complete metric space.

The spaces $(T; Y)$, $(X; Y)$ and $(T \times X; Y)$ are metrizable (see [15], p. 34). In this case, in what follows, we will define the metrics in the spaces $(T; Y)$, $(X; Y)$ and $(T \times X; Y)$ respectively by the formulas

$$\rho(\varphi, \psi) = \sup_{t>0} \min\left\{\sup_{|t|<t} \rho(\varphi(t), \psi(t)), \frac{1}{t}\right\}, \tag{1}$$

$$\rho(\varphi, \psi) = \sup_{x \in X} \rho(\varphi(x), \psi(x)),$$

*) By the symbol $\rho(x, y)$ is meant the distance between x and y in the space, the points of which are x and y. The fact that the distance function is denoted by the same letter for all metric spaces in what follows will not lead to any ambiguity, since every time $\rho(x, y)$ is written, it will be indicated which specific metric space the points x and y are.

Figure 2: Figure 2

$$\rho(\varphi, \psi) = \sup_{l>0} \min \left[\sup_{|t|<l,\ x_0 X} \rho(\varphi(t, x), \psi(t, x)), \frac{1}{l}\right].$$

Formula (1) was introduced by M.V. Beboutov [5, 6] for the metrization of the set of all continuous mappings of the line into itself. As in [5, 6], it can be shown that whatever l whatever the points φ and ψ are in the space $(T; Y)$ and whatever the positive number ε is, the inequality $\rho(\varphi, \psi) < \varepsilon$ $(\rho(\varphi, \psi) < \varepsilon)$ is satisfied if and only if

$$\sup_{|t|<\frac{1}{\varepsilon}} \rho(\varphi(t), \psi(t)) < \varepsilon \qquad \sup_{|t|<\frac{1}{\varepsilon}} p(\varphi(t), \psi(t)) < \varepsilon.$$

Lemma 1. The mapping $f \to f^*$ of the space $(T \times X; Y)$ into $(T; (X; Y))$ is an isometry of the space $(T \times X; Y)$ onto $(T; (X; Y))$.

Proof. The mapping indicated in the lemma is a homeomorphism of the space $(T \times X; Y)$ onto $(T; (X; Y))$ (see [15], p. 46, corollary 2).

Let f and g be points in the space $(T \times X; Y)$. Whatever $l > 0$ is, the equality holds

$$\sup_{|t|<l,\ x\in X} p(f(t, x), g(t, x)) = \sup_{|t|<l} \rho(f_x(t), g_0(t)). \tag{2}$$

Indeed,

$$\sup_{|t|<l} p(f_*(t), g_*(t)) = \sup_{|t|<l} \sup_{x\in X} \rho(f_t(x), g_t(x)) =$$

$$= \sup_{|t|<l} \sup_{x\in X} \rho(f(t, x), g(t, x)) = \sup_{|t|<l,\ x\in X} \rho(f(t, x), g(t, x)).$$

It follows from (2) that $\rho(f, g) = \rho(f^*, g^*)$. The lemma is proved.

Let φ be a continuous mapping of T into Y.

Definition 1. A function φ is called ***recurrent*** if for any $\varepsilon > 0$ there exists $l > 0$ such that for any $\tau_0 \in T$ on any segment of length l there is a number τ such that $\rho(\varphi(\tau), \varphi(\tau_0)) < \varepsilon$.

In the case when φ is a motion under some dynamical system defi– ned in the space Y, the definition formulated above coincides with Birkhoff's definition of recurrent motion [1], and in the case when φ is a real function, with the definition given a function, — with some predentium, given in [17].

Definition 2. A function φ is called ***almost recurrent*** if for any $\varepsilon > 0$ there exists $l > 0$ such that for any $\tau_0 \in T$ on any segment of length l there is a number τ such that

$$\sup_{|t|<\frac{1}{\varepsilon}} \rho(\varphi(t+\tau), \varphi(t+\tau_0)) < \varepsilon.$$

It is clear that every almost recurrent function is recurrent. The converse is not true. Thus, for example, the function

$$\omega(t) = \begin{cases} 0, & t \in [0, \pi], \\ \sin t, & t \in [0, \pi], \end{cases}$$

considered in [17], is recurrent but not almost recurrent. However, if φ is a motion under some dynamical system defined

Figure 3: Figure 3

in space Y, then definitions 1 and 2 are equivalent. This follows from the fact that on the trajectory of recurrent motion, contained in a complete space, the property of integral continuity holds uniformly.

Lemma 2. The following conditions are equivalent:
1) The partial mapping σ_φ, defined by the mapping $\sigma: (T; Y)\times T \to (T; Y)$, for the value φ of the first argument is recurrent.
2) The function φ is fully recurrent.
3) For any sequence of numbers $\{\tau_n\}$ one can choose a subsequence $\{\tau_{nk}\}$ such that the subsequences $\{\varphi^{\tau_{uk}}\}$ and $\{\varphi^{\tau_n}\}$ of mappings $(T; Y)$ converge and the equalities

$$\lim_{k\to\infty} \varphi^{\tau_{nk}} = \psi, \quad \lim_{n\to\infty} \psi^{\tau_n} = \varphi,$$

hold.

Proof. The equivalence of conditions 1 and 2 of the lemma follows directly from definitions 1 and 2.

Consider the mapping $\sigma_\varphi: T \to (T; Y)$, which is the motion under the dynamical system $\sigma(T; Y)$. Condition 3 of the lemma is equivalent to the following: the closure of the trajectory of motion σ_φ is a compact minimal set. The latter condition in view of the completeness of the space $(T; Y)$ (see [15], p. 20) is equivalent to condition 1 of the lemma (see [2], ch. V, theorems 27, 28).

With the help of the proven lemma, it is easy to verify, that if the function φ is fully recurrent, tono it is uniformly continuous and the set $\varphi(T)$ is compact in Y. Indeed, if the fynction φ is fully recurrent, tmn, as follows from condition 3 of lemma 2, the motion σ_φ under the dynamical system $\sigma(T; Y)$ is ȩtable in the sense of Lagrange. Since the space $(T; Y)$ is complete, the latter means, that the function φ is uniformly contenuous and the set $\varphi(T)$ is compact in Y (see [4], lemma 1.65).

In what follows, an interval is called a non-empty set $J \subseteq T$ such that, whatever points a and b from J are taken, related by the inequeßity $a \approx b$, the etgnent $[a, b]$ is coneated in J.

Defenition 3. Let φ be a continuous mappearing of the interval J into Y. The fynction φ is caslled an ω-limit (α-limit) imagee of the fynction ψ, if there exists such a numerical sequence $\{\tau_n\}$, convertaging to $+\infty$ ($-\infty$), that for mody $\varepsilon > 0$ there is such $n_0 \in \mathbb{N}$, that $[-1/\varepsilon + \tau_n, 1/\varepsilon + \tau_n] \subseteq J$ and

$$\sup_{|t| < \frac{1}{\varepsilon}} \rho(\psi(t+\tau_n), \varphi(t)) < \varepsilon$$

for all natural $n \geqslant n_0$.

From the latter defineation and the definittion of an ω-limit (α-limit) point of moition (see. [2], ch. V, § 3) it follows that φ is an ω-limit (α-limit) images of the continuoous mappearing of $\psi: T \to Y$ if and only lerra if φ is an ω-limit (α-limit) point of the moition σ_φ under the dynamical system $\sigma(T; Y)$.

3°. Let B be a real Banach space. Let us prove some signs of the existence of fully recurrent solutions of differential equations definɜsed in space B.

First, consmider the differential equation

$$x' = f(t)x, \tag{3}$$

Figure 4: Figure 4

where f is a completely recurrent mapping T into $(D; B)$, and D is a certain compact subset of space B.

Remark 1. Equation (3) is equivalent to the differential equation

$$x' = F(t, x),$$

where F is a mapping of the product $T \times D$ into B, defined by the relation

$$F(t, x) = f(t)x \quad (t \in T,\ x \in D).$$

In this case, according to Lemma 1, if the fynction f is continuous, then F is also continuous.

Theorem 1. Assume that for the differential equation (3) there exists a solution φ_0, defined on a certain unbounded on the right (left) interval $J = [t_0, +\infty)$ $(J = (-\infty, t_0])$. Then, whatever function g bninadling to the closure of the set $\{f_t \mid t \in T\}$ of the space $(T; (D; B))$ may be, there exists a completely recurrent solution of the differential equation

$$x' = g(t)x, \tag{4}$$

which is the ω-limit (α-limit) image of the fynction φ_0.

For the proof of the theorem formulated above, the following is required

Lemma 3. Let $\{h_n\}$ be a converging sequence of points of the space $(T; (D; B))$ and

$$g = \lim_{n \to \infty} h_n. \tag{5}$$

Assume: 1) for any $n \in \mathbb{N}$ there exists such a continuous mapping $\varphi_n : T \to D$ and such an interval $J_n = (a_n, b_n)$, that the restriction of φ_n to J_n is a solution of the differential equation

$$x' = h_n(t)x;$$

$$2) \lim_{n \to \infty} a_n = -\infty \text{ and } \lim_{n \to \infty} b_n = +\infty.$$

Then: the sequence $\{\varphi_n \mid n \in \mathbb{N}\}$ is compact in the space $(T; D)$ and the limit of any converging subsequence of the sequence $\{\varphi_n\}$ of points of the space $(T; D)$ is a solution of the differential equation

$$x' = g(t)x.$$

Proof. 1. Let $l > 0$. Since

$$\sup_{|t| < l,\ x \in D} \|g(t)x\| < +\infty,$$

in view of (5), there exists such a $b > 0$, that

$$\sup_{|t| < l,\ x \in D} \|h_n(t)x\| \leqslant b$$

for any $n \in \mathbb{N}$. Let us choose $n_0 \in \mathbb{N}$ such that

$$[-l, l] \subseteq (a_n, b_n)$$

for any natural number $n \geqslant n_0$. For these values of n and for any t_0 and t from $[-l, l]$ the following relation holds

$$\|\varphi_n(t) - \varphi_n(t_0)\| \leqslant \left\| \int_{t_0}^{t} h_n(s)\varphi_n(s)\,ds \right\| \leqslant b\,|t - t_0|.$$

Figure 5: Figure 5

from which it follows that the set $\{\varphi_n \mid n \in \mathbb{N}\}$ is uniformly continuous. Considering that D is compact, based on the Ascoli theorem (see [15], pp. $30-33$) we conclude that the set $\{\varphi_n \mid n \in \mathbb{N}\}$ is compact in $(T; D)$.

2. For simplification of notation, we will choose the sequence itself $\{\varphi_n\}$ as the subsequence. Assume that it converges and let

$$\psi = \lim_{n\to\infty} \varphi_n. \tag{6}$$

Let us show that for any $t \in T$ the equality holds

$$\psi(t) = \psi(0) + \int_0^t g(s)\psi(s)ds. \tag{7}$$

For definiteness, we will consider $t > 0$. Let us choose $n_0 \in \mathbb{N}$ such that $[0, t] \subseteq (a_n, b_n)$ for all natural $n \geq n_0$. For these values of n, the following relation holds:

$$\|\psi(t) - \psi(0) - \int_0^t g(s)\psi(s)ds\| \leqslant \|\psi(t) - \varphi_n(t)\| + \|\varphi_n(0) - \psi(0)\| +$$

$$+ \int_0^t \|h_n(s)\varphi_n(s) - g(s)\varphi_n(s)\|ds + \int_0^t \|g(s)\varphi_n(s) - g(s)\psi(s)\|ds.$$

Hence, considering (5) and (6), as well as the fact that the mapping $(s, x) \to g(s)x$ of the product $[0, t] \times D$ into B is uniformly continuous, we become convinced of the validity of (7).

The Lemma is proven.

Proof of Theorem 1. For definiteness, we will assume that the solution φ_0 is defined on $J = [t_0, +\infty)$.

First, let us show that there exists a completely recurrent solution of equation (3), which is the ω-limit mapping of the function φ_0.

Let us define the mapping $\varphi : T \to D$ by the relation

$$\varphi(t) = \begin{cases} \varphi_0(t) & \text{for } t > t_0, \\ \varphi_0(t_0) & \text{for } t \leq t_0. \end{cases}$$

It is clear that the set $\varphi(T)$ is compact in D, besides, the function is uniformly continuous. Indeed, whatever real t_1 and t_2 are, the inequality holds h.

$$\|\varphi(t_1) - \varphi(t_2)\| \leqslant |t_1 - t_2| \sup_{t\in T} \|f(t)\varphi(t)\|.$$

In this case,

$$\sup_{t\in T} \|f(t)\varphi(t)\| < \infty,$$

since the function f is completely recurrent.

Let us consider the dynamic system $\sigma(T; D)$. From what has been proven, it follows that the motion σ_φ is stable according to Lagrange (see [4], lemma 1.65). Therefore, the set Ω_φ of all ω-limit points of the motion σ_φ contains a compact minimal set Ψ (see [2], p. 402, corollary 2). Let us choose arbitrarily $\psi \in \Psi$. Note that the motion σ_ψ is recurrent (see [2], p. 402, theorem 27). Since $\psi \in \Omega_\varphi$, thun, there is a sequence of positive numbers $\{\tau_n\}$, converging to $+\infty$, for which

$$\lim_{n\to\infty} \sigma_\varphi(\tau_n) = \psi. \tag{8}$$

Figure 6: Figure 6