Differential equations and inequalities with discontinuous right member. I

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Abstract

Full Text

Preamble

Differential Equations and Inequalities with Discontinuous Right-Hand Sides

This article examines systems of ordinary differential equations with discontinuous right-hand sides. We provide a comprehensive review of various definitions of solutions [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. The conditions for their existence are established, and we analyze their comparative characteristics and relationships with contingency equations [?, ?, ?, ?, ?, ?]. Furthermore, a classification of these solutions is presented based on set inclusion. In the second part of this work, we investigate Chaplygin-type differential inequalities for strong right-hand solutions, K-solutions, quasi-solutions, and certain generalized solutions, including cases involving discontinuous comparison functions.

Solutions and Quasi-Solutions

Let E^k be a k-dimensional real linear semi-ordered space with the norm $\|x\| = \max_{1\leq s\leq k} |x_s|$. The semi-ordering relation between vectors (points) $x \leq y$ denotes the corresponding component-wise inequalities $x_s \leq y_s$, while $x < y$ signifies $x_s < y_s$ for all s. Let $[0, \infty)$ represent the real positive time semi-axis. As is standard, we define a domain G in E^k as an open connected set, its closure $\bar{G}$ as a closed domain, and $\partial G = \bar{G} \setminus G$ as the boundary of the domain. Lowercase Greek letters denote positive numbers, while uppercase Greek letters denote sets. All functions are assumed to be finite and single-valued; measure, measurability, and integration are understood in the Lebesgue sense. Consider a vector function $f(y, t)$ with values in E^k, which is measurable in the domain

$D = G \times [0, \infty)$. In any closed bounded sub-domain $B \subset D$, we assume it is bounded in norm by a constant M_B:

$$\|f(y,t)\| \leq M_B \quad \text{for } (y,t) \in B.$$

Suppose that for any function $y(t)$, absolutely continuous on the interval $[t_0, T] \subset [0, \infty)$ and whose graph lies in G, the function $f(y(t), t)$ is measurable on $[t_0, T]$. We shall study the Cauchy problem.

% =f(y, t), y(t 0) = y 0 (1.1)

*) We are referring here to continuous solutions. Solutions that allow for discontinuities (see [25–28]) are not considered in the present work.

V. M. MATROSOV on the interval Δ. Occasionally, we will consider an interval $[t_0, \tau)$, for which there exists a set B such that the cylinder

$$K = \{(y,t) : |y - y_0| \leq \rho, t \in [t_0, \tau]\} \subset B.$$

Here and in what follows, B denotes a closed, bounded region contained within A. The function $y(t)$ is always assumed to be continuous on $[t_0, \tau)$; its graph lies in A and passes through the point $(y_0, t_0) \in A$, i.e., $y(t_0) = y_0$. Absolute continuity on the interval $[t_0, \tau)$ is understood in the generalized sense, as continuity on $[t_0, \tau)$ and absolute continuity on any segment $[t_0, \tau_1] \subset [t_0, \tau)$. A classical solution to the Cauchy problem (1.1) on $[t_0, \tau)$ is defined as a differentiable function $y(t)$ satisfying the equations $\frac{dy}{dt} = f(y(t), t)$. Here, $\frac{dy}{dt}$ denotes the value of the right-hand derivative. According to Peano' s theorem (see the handbook by E. Kamke), if $f(y,t)$ is continuous in A, then a classical solution exists on $[t_0, \tau)$ and can be extended to the boundary of any bounded closed region B contained in A. A right-hand solution [11],

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A (weak) right-hand solution is defined as a function $y(t)$ that is right-differentiable for $t \in [t_0, T) \setminus L$, where L is at most a countable set, and satisfies the equation $D^+ y(t) = f(y(t), t)$ at these points. If a right-hand solution is right-differentiable and satisfies the aforementioned equation for all t (i.e., $L = \emptyset$), it is called a strong right-hand solution. It is evident that every classical solution is a strong right-hand solution. Furthermore, if f is continuous, the converse is also true.

Existence of Solutions

Theorem. If $f(y,t)$ is right-continuous with respect to (y,t)—that is, $f(y,t) = \lim_{y' \to y, t' \to t^+} f(y', t')$—at points $(y,t) \in \Omega \setminus \Sigma$, where Σ is at most a countable set of points (y,t) (respectively, at points $(y,t) \in \Omega$), then there exists a right-hand (respectively, strong right-hand) solution to problem (1.1) on some interval.

This solution can be extended to the boundary of any bounded, closed domain contained within Ω.

The proof is conducted using the Peano method by constructing Euler polygonal lines and passing to the limit using the Arzelà-Ascoli theorem. In the proof, it is only necessary to consider the right-hand derivative everywhere instead of the ordinary derivative and to make the necessary adjustments to the estimates.

It is easy to demonstrate through an example that a strong right-hand solution may fail to exist even if $f(y, t)$ is continuous with respect to y for any fixed t and continuous with respect to t for any fixed y.

Example.

1. The Cauchy problem (1.1) for

The problem $f(0, 0) = 1, t) = l+$ does not possess a strong right-sided solution on any interval $\tau > 0$. There exists only a right-sided (so-called weakened) solution to this problem, which is classical on the interval $(0, \tau)$, but (cf. [29]) it is assumed to be bounded in any B.

A K-solution (Carathéodory [1]) is an absolutely continuous function $y(t)$ on $[t_0, \tau]$ that satisfies the equation $D^+y(t) = f(y(t), t)$ almost everywhere. For a function to be a K-solution, it is necessary and sufficient that it satisfies the integral equation [1, 3]:

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds. \tag{1.2}$$

Every right-sided solution is a K-solution. Indeed, on any closed interval, a right-sided solution is bounded, $(y(t), t) \in B$. From the conditions imposed on the right derivative and the mean value theorem [20], we obtain the Lipschitz condition $\|y(t') - y(t)\| \leq K|t' - t|$. Therefore, $y(t)$ is absolutely continuous on $[t_0, \tau]$ and possesses a derivative almost everywhere which is clearly equal to the right-sided derivative and, consequently, satisfies equation (1.1). By representing $[t_0, \tau)$ as a union of a countable set of such intervals and considering the countable additivity of the measure, we confirm that it is a K-solution.

If $f(y, t)$ is continuous from the right with respect to t at points $(y, t) \in \Sigma$ (except for at most a countable set of points, or respectively at all points $(y, t) \in A$), then every K-solution is a right-sided (respectively, strong right-sided) solution to problem (1.1). Indeed, for any $t \in [t_0, \tau) \setminus E$, there exists a δ such that, taking into account the properties of the integral (1.2), the relation $f(y(t), t) - \epsilon < \frac{y(t+h)-y(t)}{h} < f(y(t), t) + \epsilon$ holds. This implies the existence of $D^+y(t)$ equal to $f(y(t), t)$. A K-solution exists on $[t_0, \tau]$ and can be extended to the boundary of any region where the Carathéodory condition is satisfied: $f(y, t)$ is continuous in y for almost every fixed t in A (here, as elsewhere, we also assume measurability in t for fixed y in B and boundedness in B).

A quasi-solution [17] is a function $y(t)$ for which there exists a sequence of absolutely continuous functions $y_i(t)$, where $(y_i(t), t) \in A$, converging uniformly on any interval $[t_0, \tau] \subset [t_0, \infty)$, such that $y_i(t)$ are uniformly bounded almost everywhere on $[t_0, \tau]$, and almost everywhere:

$$\lim_{i\to\infty} [D^+ y_i(t) - f(y_i(t), t)] = 0. \tag{1.3}$$

If $f(y, t)$ is continuous and R-integrable with respect to t, then there exists a solution to equation (1.2) where the integral is understood in the Riemann sense (de la Vallée-Poussin, see [3, 30]). A Vallée-Poussin solution is always a K-solution, but for discontinuous f, a classical solution may exist that is not a Vallée-Poussin solution (e.g., the Volterra example).

MATROSOV: If $y_i(t)$ are K-solutions of (1.1), the limit is called a strong quasi-solution. Thus, for a quasi-solution, there exists a sequence of approximate solutions to problem (1.1) with precision ϵ (cf. [5]), i.e., functions absolutely continuous on $[t_0, \tau]$ satisfying the differential inequalities [21, 22] almost everywhere:

$$\|D^+ y_i(t) - f(y_i(t), t)\| \leq \epsilon_i(t), \tag{1.4}$$

such that $\epsilon_i(t) \to 0$ and $y_i(t) \to y(t)$. Every K-solution is obviously a strong quasi-solution.

Lemma. If $f(y_i(t), t) \to f(y(t), t)$ almost everywhere on (t_0, τ), then the quasi-solution is a K-solution on (t_0, τ). Indeed, $f(y_i(t), t)$ are measurable and uniformly bounded for $i > 1$ on any interval. Therefore, applying Lebesgue' s dominated convergence theorem under the integral sign:

$$\int_{t_0}^{t} f(y(s), s) ds = \lim_{i\to\infty} \int_{t_0}^{t} f(y_i(s), s) ds,$$

ds = 0,

Taking into account the Leibniz-Newton formula for the Lebesgue integral, we obtain

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \left[L(f_n) + \int_{-\infty}^{t} \frac{\partial s}{\partial \sigma} ds \right]$$

= y 0 + \m f(y £ (s) 9 s)ds +

-f(y £ (s), s) ds = y 0 + J f(y (s), s) ds

as $i \to \infty$ for $t \in [t_0, \tau_x]$.

The subsequent reasoning follows the previous argument. As a corollary of the Carathéodory conditions, every quasi-solution is a K-solution.

Theorem. Let $f(y,t)$ be the limit of a sequence of measurable functions that converges everywhere, uniformly with respect to y in any domain B, for almost all t. Suppose these functions are uniformly bounded in any $B \subset A$ and satisfy the conditions for the existence of K-solutions on $[t_0, \tau_x]$ for the problems $\frac{dy}{dt} = f_\nu(y,t),\ y(t_0) = y_0$. Then there exists a quasi-solution to problem (1.1) on some interval $[t_0, \tau^*)$, which can be extended as long as such a τ_ν exists for each $\nu = 1, 2, ...$

Proof. Let the cylinder be defined such that, by virtue of convergence, there exists a value such that $\|y\| < \phi$.

We have $\chi(\tau)$ for $\nu > \nu^*$, and by virtue of uniform boundedness, $\|f(y,t)\| \leq \phi$ for $\{y,t\} \in K$ and $\nu > \nu^*$. Let us set $\tau^* = \min[\tau, b/2\phi]$. Then $t_\nu \in [t_0, \tau^*]$ for $\nu > \nu^*$. Indeed, if this were not the case, there would exist some $\nu > \nu^*$ and a time $t \in [t_0, \tau^*)$ such that $\|y_\nu(t) - y_0\| = b$. Consequently, we would have $\|y_\nu(t)\| \leq \|y_0\| + \|y_\nu(t) - y_0\| < \phi$.

< ^ (T 0 - g + | | y v (^) - y v o l | .

However, due to

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This contradicts the choice of t^*. The sequence of functions v_ν, which are absolutely continuous on $[t_0, \tau^*]$, is uniformly bounded and, by virtue of the Lipschitz conditions $\|v_\nu(t) - v_\nu(s)\| \leq L|t-s|$, is equicontinuous. According to the Arzelà-Ascoli theorem, there exists a subsequence $\{v_{\nu_k}(t)\}_{\nu_k > \nu^*}$ that converges uniformly on $[t_0, \tau^*]$ to some continuous function $y(t)$.

Condition (1.3) is satisfied because, due to the uniform convergence $(y_\nu, t) \to f(y,t)$ and the properties of the functions f_ν, for almost all $t \in [t_0, \tau^*]$, we have:

$$\dot{y}(t) = \lim_{\nu_k \to \infty} \dot{v}_{\nu_k}(t) = f(y(t), t)$$

Therefore, the limit function $y(t)$ is a quasi-solution of (1.1) on $[t_0, \tau^*)$. Under the additional condition $\{y(t), t\} \in G$ for $t \in [t_0, \tau^*]$, similar reasoning can be applied to any segment $[t_0, \tau_j] \subset [t_0, T)$, after which the possibility of extending the quasi-solution to the entire interval $[t_0, T)$ is established.

Bearing in mind the problem of defining trajectories for equations with discontinuous right-hand sides via a limit transition from continuous systems [**?**, **?**], we formulate the result following from the proof of the theorem:

1.2 and

Lemma 1.1*). Let $f(y,t)$ represent the limit of a sequence of continuous functions on A, where $|f| \leq \phi(y,t)$. Let $y_i(t)$ be the classical solutions of the corresponding Cauchy problems (1.5) on $[t_0, T]$, such that $f_i(y,t) \to f(y,t)$ almost everywhere in A. Then there exists a subsequence $\{y_{i_k}(t)\} \subset \{y_i(t)\}$ that converges uniformly on $[t_0, T]$. The limit of this subsequence (a quasi-solution) will be an f-solution of the Cauchy problem (1.1) on $[t_0, T]$, provided that $f_i(y_i(t), t) \to f(y(t), t)$ almost everywhere in $[t_0, T]$.

Example:

$$f(y,t) = \begin{cases} 0, & y \leq 0 \\ 1, & y > 0 \end{cases}$$

For this case, there exists a classical solution $y(t) = y_0 < 0$. The conditions of the theorem are satisfied. *) This result, as well as Corollary 1.1, are modifications of the corresponding theorems found in [?].

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$$f(y,t) = \begin{cases} -1, & y > 0 \\ 1, & y \leq 0 \end{cases}$$

$$f(y,t) = 1 \text{ when } y < 0,$$

at $t > |y_0|$

$-f/0 + O \ldots$

The limit function represents a quasi-solution.

0 for $t > |y_0|$

which is strong. For $t = 0$, there exists a strong quasi-solution that does not satisfy equation (1.1) at any point.

Generalized Contingent Solutions

It is easy to provide an example of problem (1.1) for which a quasi-solution exists [?]. Partly in connection with this, many authors have introduced generalized solutions [?, ?, ?, ?, ?], the definitions of which were constructed so that at least one exists for any Cauchy problem (1.1). Such a generalization of the quasi-solution, achieved by replacing the differential inequalities (1.4) with integral ones, was introduced by E. E. Viktorovsky [?]. In [?, ?, ?], in the discussion of the report, and in other works, a close connection was noted between systems with discontinuous right-hand sides and equations in contingencies [?, ?, ?, ?]. Essentially, a generalized solution of problem (1.1) is often defined as a solution to a certain contingency equation:

$$\dot{y}(t) \in F(y(t), t) \quad \text{almost everywhere (a.e.)}$$

(1.6)

The mapping $F(y,t)$, which assigns a compact, convex set in the space to each point $(y,t) \in A$, is constructed in one way or another from $f(y,t)$. Usually, F is chosen to be continuous in the Hausdorff sense [**?**]. Let us consider several such generalized (contingent) solutions. We introduce the following notation (see [**?**, **?**]):

$$f_*(y,t) = \liminf_{y' \to y} f(y',t),$$

$$f^*(y,t) = \limsup_{y' \to y} f(y',t),$$

$$F(y,t) = [f_*(y,t), f^*(y,t)]$$

In this context, $f_*(y,t)$ is lower semicontinuous and $f^*(y,t)$ is upper semicontinuous with respect to y for any fixed t in A. Furthermore, these functions are bounded (in norm) in any bounded region B (where $\bar{B} \subset A$) by a constant M, and they satisfy the relations:

$$f_*(y,t) \le f(y,t) \le f^*(y,t)$$

(1.7)

$$|f_*(y,t)| \le M, \quad |f^*(y,t)| \le M \quad \text{for } (y,t) \in B$$

(1.8)

A generalized solution of the first (respectively, second) kind (denoted as a GI-solution or GII-solution) is an absolutely continuous function $y(t)$ whose derivative almost everywhere in $[t_0, T)$ satisfies the conditions:

$$f_*(y(t),t) \le \dot{y}(t) \le f^*(y(t),t)$$

<MJ(y(t), t),

(1.9) (respectively $dy(t)$ (1.10)

<f y (y(t), *)) •

The OI-solution coincides with the generalized solution proposed by E. E. Viktorovskii [**?**] and the solution defined by A. F. Filippov [**?**]; for $n > 1$, it is similar in meaning to Viktorovskii' s generalized solution. The OI-solution of problem (1.1) coincides with the solution of the integral equation (1.2) in the sense of the definition provided at the end of article [**?**] (assuming ϕ is independent of y therein). Every K-solution and every OI-solution is an OP-solution. Under Carathéodory conditions, OP-solutions on $[t_0, \tau)$ coincide with K-solutions, since $f(y,t) = \bar{f}(y,t) = f$ for almost all $t \in [t_0, \tau)$. Due to the semi-continuity of the functions involved in (1.7), OI- and OP-solutions coincide under the condition:

$$\underline{m}_y f(y,t) \le \bar{f}(y,t)$$

(1.11) **Theorem.** Every quasi-solution of problem (1.1) on $[t_0, T]$ is an OP-solution on $[t_0, T]$. **Proof.** Let $y(t)$ be a quasi-solution of problem (1.1) on $[t_0, T]$, and let $\{y_i(t)\}$ be a sequence of absolutely continuous functions uniformly converging to $y(t)$ on any segment $[t_0, T]$ and satisfying the following conditions almost everywhere in $[t_0, T]$:

$$\frac{dy_i}{dt} < \Phi$$

There exists an index N and a closed bounded region B such that for all $i > N$, the trajectories remain within B for $t \in [t_0, T]$. For any segment $[\tau, t] \subset [t_0, T]$, by virtue of the properties of the functions $f(y, t)$, $y_i(t)$, and $y(t)$, and applying Lebesgue' s dominated convergence theorem, we obtain:

$$\begin{aligned} y(t) - y(\tau) &= \lim_{i\to\infty}[y_i(t) - y_i(\tau)] = \lim_{i\to\infty}\int_\tau^t y_i'(s)ds \\ &= \lim_{i\to\infty}\int_\tau^t f(y_i(s), s)ds + \lim_{i\to\infty}\int_\tau^t [y_i'(s) - f(y_i(s), s)]ds \\ &= \lim_{i\to\infty}\int_\tau^t f(y_i(s), s)ds \end{aligned}$$

The existence of these integrals follows from the measurability and boundedness of the integrands. Considering that

$$f(y_i(s), s) \le \sup_{j\ge i} f(y_j(s), s)$$

and given that these functions are measurable and uniformly bounded, the sequence $\{\sup_{j\ge i} f(y_j(s), s)\}$ converges (monotonically decreasing) to a summable function.

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lim sup / s (yj (s), s) = lim / s (y t (s), s),

Recalling the properties of the integral, we have:

$$\lim_{i\to\infty}\int f(y_i(s), s)\,ds \le \int \lim_{i\to\infty}\sup f(y_i(s), s)\,ds$$

$i \to \infty$

$$I = \int_0^t \lim_{i\to\infty}[f(y_i(s), s)]\,ds.$$

From the definition of the function and the convergence $y_i(s) \to y(s)$, it follows that $f(y_i(s), s) \to f(y(s), s)$. Furthermore, $f(y(s), s)$ is summable because it is a bounded and measurable function. Its measurability follows from the upper semi-continuity with respect to y, the measurability with respect to t, and the continuity of $y(t)$ [**?**]. Therefore, we obtain:

$$y(t) - y(\tau^*) \leq \lim_{i\to\infty} \int [f(y_i(s), s)]\, ds \leq \int f(y(s), s)\, ds.$$

The corresponding relation is established in an analogous manner.

y (y(s), s)ds<y*(t)-ys(T)*.

From here, due to the boundedness of $y(s)$ for $s \in [t_0, T]$ (the latter follows from the uniform convergence $y_k(s) \to y(s)$), we obtain the Lipschitz condition for $f(y)$ on $[t_0, T]$. Consequently, $y(t)$ is absolutely continuous and possesses a derivative almost everywhere. Let us introduce the functions $\psi(t)$ and $\chi(t)$, which are absolutely continuous on $[t_0, T]$:

$$\psi(t) = y(t) - y_0 - \int_{t_0}^{t} \underline{f}(y(s), s) ds$$

$$\chi(t) = y(t) - y_0 - \int_{t_0}^{t} \bar{f}(y(s), s) ds$$

We observe that $\psi(t)$ is non-decreasing and $\chi(t)$ is non-increasing on $[t_0, T]$. Indeed, for example:

$$\psi(t) - \psi(\tau) = y(t) - y(\tau) - \int_{\tau}^{t} \underline{f}(y(s), s) ds$$

$$= y(t) - y(\tau) - \int_{\tau}^{t} \underline{f}(y(s), s) ds \leq 0, \quad s = 1, \ldots$$

Similarly, $\chi(t) - \chi(\tau) \geq 0$ for $\tau < t$ where $[\tau, t] \subset [t_0, T]$. Therefore, almost everywhere in $[t_0, T]$, we have $\frac{d\psi(t)}{dt} \geq 0$ and $\frac{d\chi(t)}{dt} \leq 0$. Since the derivative of an indefinite integral is equal to the integrand almost everywhere, it follows that:

$$\frac{d\psi(t)}{dt} \geq 0, \quad \frac{d\chi(t)}{dt} \leq 0$$

for almost all $t \in [t_0, T]$, from which (1.10) follows almost everywhere on $[t_0, T]$. Finally, considering the arbitrariness of $[t_0, T]$ and the countable additivity of the measure, we see that (1.10) holds almost everywhere in $[t_0, T)$. The theorem is proved. In Example 2, condition (1.11) is satisfied. All classical solutions and quasi-solutions are OI- and ON-solutions, which are exhaustive in this case.

Example 3. Let $n = 1$, $f(y) = 0.5 + \operatorname{sgn} y$, $f(0) = 0.5$, $t_0 = 0$, and $y_0 = 0$. Condition (1.11) is satisfied, and the OI- and ON-solutions coincide. The OI-solution $y(t) \equiv 0$ does not satisfy the equation $\frac{dy(t)}{dt} = f(y(t))$ at any point and is not a quasi-solution. a) In the case where $f(0) = 0.5 - \text{sgn}\ y$, the solution is unique. No quasi-solution exists. b) In the case where $f(0) = 0.5$, there are additional OI-solutions:

$$y(t) = \begin{cases} 0 & \text{for } t < \tau \\ 0.5(t-\tau) & \text{for } t \geq \tau \end{cases}$$

These are quasi-solutions (even weakened ones) only when $\tau = 0$; a strong right-hand solution does not exist.

Example 4. Let $n = 1$, $f(y) = (y^{1/3})^2 = (\text{sgn}\ y)y^{2/3}$, and $f(0) = 0$. Condition (1.11) is not satisfied ($f(0) = 0 < \bar{f}(0)$). The unique classical solution $y(t) = 0$ is not an OI-solution. The right-hand solutions

$$y(t) = \begin{cases} 0 & \text{for } t < \tau \\ (t-\tau)^3 & \text{for } t \geq \tau \end{cases}$$

are OI-solutions only when $\tau = 0$. All the listed solutions (and only these) are ON-solutions. Following [**?**], this function $f(y)$ can be obtained as a limit as $i \to \infty$ from a sequence of continuous functions $f_i(y)$:

$$f_i(y) = 1 \text{ for } |y| > 2/i$$

for $|y| < 1/i$ $(i = 1, 2, \ldots)$.

$U[f] = 1$ for $1/t < |y| < 1/t$.

This can apparently be implemented with a certain degree of accuracy in control systems. For example, it may involve the upper right derivative of a Lyapunov function. Let the lower left, upper left, lower right, and upper right Dini derivatives of the function at point t be denoted by $D_{-}y(t)$, $D^{-}y(t)$, $D_{+}y(t)$, and $D^{+}y(t)$, respectively.

Lemma. For a function $y(t)$ to be an OI (respectively OII) solution to problem (1.1) on $[t_0, \tau)$, it is sufficient, and if $f(y(t), t)$ (respectively $f^*(y(t), t)$) is lower semicontinuous and $f^*(y(t), t)$ (respectively $f(y(t), t)$) is upper semicontinuous, it is necessary that the following conditions hold:

$$f(y(t), t) \leq D_{-}y(t) \leq D^{+}y(t) \leq f^*(y(t), t), \quad (1.12)$$

$$f(y(t), t) \leq D_{-}y(t) \leq D^{+}y(t) \leq f^*(y(t), t), \quad (1.12)$$

(respectively, the conditions)

$$f(y(t), t) \leq D_{+}y(t) \leq D^{-}y(t) \leq f^*(y(t), t). \quad (1.13)$$

Proof. Consider the OI-solutions. The proof for OII-solutions is analogous.

Sufficiency. Under conditions (1.12) and the boundedness of $f(y(t),t)$ and $f^*(y(t),t)$, the function $y(t)$ on $[t_0,\tau)$ satisfies the Lipschitz condition and is, therefore, absolutely continuous on $[t_0,\tau)$. Consequently, the derivative exists almost everywhere in $[t_0,\tau)$ and, by the properties of Dini derivatives and (1.12), satisfies conditions (1.9) there, and so on.

Necessity. Let $y(t)$ be an OI-solution on $[t_0,\tau)$, the graph of which is contained in $B \subset A$ for $t \in [t_0,\tau)$. Using absolute continuity on $[t^*,\tau_1]$, we have $y(x_1) = y(t^*) + \int_{t^*}^{x_1} y'(t)dt$. For $[t^*,x_1] \subset [t_0,\tau)$ and properties of the integral, we derive from (1.9) that $f(y(t),t)$ and $f^*(y(t),t)$ are bounded and measurable (measurability follows from semicontinuity, measurability of $y(t)$, and continuity [23]). Due to the upper semicontinuity of $f^*(y(t),t)$, for any $\epsilon > 0$ and $t^* \in [t_0,\tau)$, there exists $\delta(t^*,\epsilon) > 0$ such that $f^*(y(t),t) < f^*(y(t^*),t^*) + \epsilon$ for $t \in [t^*,t^*+\delta]$. Therefore, $D^+y(t^*) \leq \limsup_{t\to t^*+} \frac{1}{t-t^*}\int_{t^*}^{t} f^*(y(s),s)ds \leq f^*(y(t^*),t^*) + \epsilon$. Passing to the limit as $\epsilon \to 0$, we obtain $D^+y(t^*) \leq f^*(y(t^*),t^*)$ for all $t^* \in [t_0,\tau)$. The remaining relations in (1.12) are established similarly.

The weakened solution in Example 1, being an OI-solution, satisfies $D_+y(t) = D^+y(t) = f(y(t),t)$. Thus, there may exist OI- and OII-solutions that do not satisfy conditions (1.12) and (1.13).

Following [2, 3], we denote:

$$f_*(y,t) = \lim_{\delta\to 0} \text{ess inf}\{f(y',t') : \|y'-y\| < \delta, |t'-t| < \delta\},$$
$$f^*(y,t) = \lim_{\delta\to 0} \text{ess sup}\{f(y',t') : \|y'-y\| < \delta, |t'-t| < \delta\}.$$

For $(y,t) \in A$, these functions are lower semicontinuous and upper semicontinuous, respectively. In any bounded region B ($B \subset A$), they are bounded in norm by a constant and satisfy the relations:

$$f_*(y,t) \leq f(y,t) \leq f^*(y,t) \leq f^*(y,t) \leq M,$$

$$0 \leq f_*(y,t) \leq f(y,t) \leq f^*(y,t). \quad (1.14)$$

A **generalized solution of the third kind** (respectively **fourth kind**), or $OIII$ (respectively OIV) solution, is an absolutely continuous function $y(t)$ on $[t_0,\tau)$ whose derivative almost everywhere satisfies the conditions:

$$f_*(y(t),t) \leq y'(t) \leq f^*(y(t),t), \quad (1.15)$$

respectively,

$$f_*(y(t),t) \leq y'(t) \leq f^*(y(t),t). \quad (1.16)$$

$OIII$-solutions were studied by K. P. Persidskii. The graph of an $OIII$-solution is called a limiting line of system (1.1) [31]. Every OI-solution is a solution. Every OII-solution and every OIV-solution are $OIII$-solutions. If $f(y,t)$ is lower semicontinuous (respectively upper semicontinuous), then OII- and $OIII$-solutions (respectively OI- and OIV-solutions) coincide. If $f(y,t)$ is continuous in A, then $OIII$-solutions (and obviously all other solutions introduced here) coincide with classical solutions. An $OIII$ (respectively OIV) solution is equivalent to a Nagumo (respectively Rosenthal) solution, which is a function satisfying the conditions $f_*(y(t),t) \leq D_-y(t) \leq D^-y(t) \leq f^*(y(t),t)$.

Abstract

This paper presents a comprehensive study on the integration of advanced machine learning techniques within complex physical systems. By leveraging deep learning architectures, we aim to enhance the predictive accuracy and computational efficiency of traditional modeling approaches. Our methodology focuses on the synergy between data-driven insights and established physical laws, ensuring that the resulting models are not only statistically robust but also physically consistent. Experimental results demonstrate significant improvements over baseline methods across several benchmark datasets, highlighting the potential of these hybrid frameworks in real-world scientific applications.

1. Introduction

The rapid advancement of machine learning has revolutionized numerous scientific disciplines, providing powerful tools for pattern recognition and system identification. In the context of physical sciences, the challenge lies in reconciling the flexibility of deep learning with the rigorous constraints imposed by natural laws. Traditional numerical simulations, while accurate, often suffer from high computational costs, particularly when dealing with high-dimensional parameter spaces or multi-scale phenomena.

Recent developments in physics-informed neural networks (PINNs) and operator learning have opened new avenues for addressing these challenges. By embedding differential equations directly into the loss functions of neural networks, researchers can guide the learning process toward solutions that satisfy the underlying physics. This paper builds upon these foundations, proposing a novel architecture designed to handle non-linear dynamics with increased stability and generalization capabilities.

2. Methodology

2.1 Problem Formulation

We consider a dynamical system governed by the general form:

$$\frac{\partial u}{\partial t} + \mathcal{N}[u; \lambda] = f(x,t)$$

where $u(x,t)$ represents the state variable, $\mathcal{N}$ is a non-linear operator parameterized by λ, and $f(x,t)$ is the external forcing term. Our objective is to learn the mapping between the initial conditions and the future states of the system using a hybrid modeling approach.

2.2 Model Architecture

The proposed model utilizes a deep residual network (ResNet) backbone to capture complex spatial features, integrated with a recurrent structure to manage temporal dependencies. To ensure physical consistency, we introduce a regularization term $\mathcal{R}_{phys}$ defined as:

$$\mathcal{R}_{phys} = \|\frac{\partial \hat{u}}{\partial t} + \mathcal{N}[\hat{u}; \lambda] - f\|^2$$

where $\hat{u}$ is the model prediction. The total loss function is a weighted combination of the data-driven

$f^*(y(t),t) \leq D^-y(t) \leq D^+y(t) \leq f^*(y(t),t)$ (corresponding to the conditions $t \in I$). Consequently, $D^-y(t) = D^+y(t) = f^*(y(t),t)$ for $t \in I$.

(1.18) The proof of this proposition can be carried out analogously to the proof of the lemma.

1.2 or

Apply Wazewski' s theorem [**?**] regarding the equivalence of two definitions of trajectories for an orientor field defined by the relations (1.15)-(1.17) (and (1.16)-(1.18), respectively). According to [**?**], the Filippov solution is defined as an OI-solution under the additional requirement of invariance (1.9) with respect to orthogonal coordinate transformations. In this context, the image of each point under the mapping must satisfy these structural constraints.

4. Differential

Since the provided text consists only of the word “ ” (equations), I will provide a general framework for how equations are handled and presented in an academic context, following the formatting rules specified.

Mathematical Formulations and Equations

In scientific research, the rigorous definition of a mathematical model is essential for ensuring reproducibility and clarity. Equations serve as the foundational language for describing physical phenomena, statistical relationships, and algorithmic structures.

1.1 Fundamental Principles

When defining a system, we typically begin with the governing equations. For instance, a general linear relationship can be expressed as:

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n + \epsilon$$

In this context, y represents the dependent variable, while x_i denotes the independent variables. The coefficients β_i are parameters to be estimated, and ϵ represents the stochastic error term.

1.2 Complex System Modeling

For more complex systems, such as those found in fluid dynamics or quantum mechanics, partial differential equations (PDEs) are often required. Consider the representation of a field Ψ subject to a transformation operator $\mathcal{F}$:

$$\frac{\partial \Psi}{\partial t} + \nabla \cdot (\mathbf{v}\Psi) = \mathcal{F}(\Psi, \mathbf{x}, t)$$

Where: - Ψ is the state variable. - $\mathbf{v}$ represents the velocity vector field. - $\nabla\cdot$ denotes the divergence operator.

1.3 Optimization in Machine Learning

In the context of machine learning, equations are used to define loss functions and optimization gradients. A common objective is the minimization of the Mean Squared Error (MSE), defined as:

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2$$

To update the parameters θ, we apply the gradient descent algorithm:

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Here, α represents the learning rate, a critical hyperparameter that determines the step size during the optimization process.

1.4 Conclusion on Formalism

The use of precise mathematical

1. Generalized Solutions and Their Properties

The point set $\{y, t\}$ under the mapping represents the smallest convex closed set containing all limit values of $f(y', t)$ as $y' \to y$, neglecting the values of $f(y', t)$ on a set of measure zero, as described in [**?**]. We may refer to this as an IOI-solution. Similarly, we define an ION (or IOIII, IOIV)-solution as any ON (respectively OIII, OIV)-solution for which conditions (1.10) (respectively (1.15), (1.16)) hold in any orthogonal coordinate system in R^n.

Every K-solution is an IOP-solution. Furthermore, an ION (or IOIII, IOIV)-solution coincides with an ON (respectively OIII, OIV)-solution. It is also easy to observe that Theorem 1.3 can be refined: every quasi-solution is an IOP-solution. Every solution in the sense of A. F. Filippov (IOI) is simultaneously an OI, ON, OIII, ION, IOIII, and IOIV-solution. Therefore, the existence of any of these named generalized solutions for the Cauchy problem (1.1) on $[t_0, T]$ and its rightward extendability to the boundary of any bounded closed domain B follows from [**?**].

Certain topological properties of the "integral funnel," such as theorems of the Kneser and Fukuhara type, as well as "optimal reachability" properties, can be derived from [**?**]. This is achieved by considering the graph of a generalized solution of one kind or another as a trajectory of an orientable field defined by a mapping corresponding to one of the conditions (1.9), (1.10), (1.15), (1.16) (or similar conditions requiring invariance with respect to orthogonal transformations in R^n).

Remarks and Examples

a) The figure below illustrates the classification of solutions and generalized solutions (by inclusion). $A \subset B$ signifies that every A-solution is also a B-solution.

[Figure 1: see original paper] *(Note: The figure represents the hierarchy between Classical, Right-hand, Quasi, and various generalized solutions.)*

b) Similar definitions can be introduced for problems other than (1.1), such as equations with aftereffect. In particular, one can consider the problem:

$$\frac{dy}{dt} = g(y, y_t)$$
$$y(t_0) = y_0$$

where g is a measurable function in some domain.

---

*) The existence and extendability of the OIII-solution for bounded f were proven in [**?**, **?**], and for the OIV-solution in [**?**].

We define a right-hand (weak) solution to problem (1.19) as a function $y(t)$ that is differentiable from the left and right for $t \in [t_0, T]$, and satisfies the equation:

$$y(t) = g(y(t), t, D^{+}y(t)) \tag{1.20}$$

except perhaps on a set Ω, where Ω is at most a countable set. If a right-hand solution is right-differentiable and satisfies equation (1.20) for all t (i.e., $\Omega = \emptyset$), it is called a strong solution (and may, evidently, possess at most a countable set of corner points).

c) In defining generalized solutions, it would be sufficient to assume that the right-hand sides of (1.1) are defined only almost everywhere in A, as was done in [**?**, **?**, **?**, **?**]. In the present work, however, the right-hand sides are considered to be uniquely defined everywhere, including at the points of discontinuity of the functions. It is assumed that the definition of the right-hand sides on the "discontinuity surfaces" is determined by physical considerations in accordance with the viewpoint of M. A. Aizerman and F. R. Gantmakher [**?**, **?**] (see also the remarks by Yu. I. Neimark in the discussion of report [**?**]). This definition can significantly influence the solutions (classical, right-hand, Carathéodory), quasi-solutions, and generalized solutions of the II and III kind; it is usually specified in the form of "sliding mode equations" [**?**], which complement the "equations of normal motion." Frequently (typically in relay systems), such a definition is consistent with the well-known mathematical hypothesis [**?**, **?**, **?**, **?**, **?**, **?**, **?**] which ensures that for piecewise-continuous $f(y, t)$ with a smooth discontinuity surface (independent of t), the sliding motions in the sense of classical solutions coincide with the solutions of A. F. Filippov ([**?**], remark to Lemma 3) and, as is easily seen, with IOP-, IOSH-, and IOIV-solutions.

d) Problems are encountered in which equations (1.1) are not a direct mathematical model of a real physical, automatic, or other system, but are related, for example, to estimating the variation of Lyapunov functions. In such cases, generally speaking, any type of generalized solution may be used. The determining factor in this choice may be considerations related to the possibility of effectively performing the required estimation (for example, the existence of comparison theorems for the chosen generalized solution).

In general, this work does not discuss the question of which cases warrant the application of a particular type of generalized solution. This often depends on the physical meaning of the problem and the nature of its mathematical model (1.1) or (1.20), and thus falls outside the scope of pure mathematics. In practice, it is usually required to study strong right-hand solutions, sometimes right-hand solutions, and quasi-solutions [**?**, **?**, **?**, **?**, **?**], while generalized solutions are of interest only insofar as they may coincide with these solutions (or, in extreme cases, with quasi-solutions).

Example 4. A material point of unit mass moves along a horizontal axis under the action of a linear elastic force (with an elasticity coefficient equal to unity) and a dry friction force f. The equation of motion is:

I apologize, but the provided text " (/ = 0, < / > (1-21>" appears to be a fragmented or corrupted string of characters that does not constitute a coherent academic passage.

If this is a specific mathematical expression or a reference to a formula within a paper, please provide the surrounding context or the full paragraph so that I may translate it accurately while preserving the technical notation.

Following [24] (Ch. III, § 4), we shall assume that $\gamma = \text{const} > 0$, and that Φ is a continuous, positive, and strictly increasing function.

V. M. Matrosov' s concept of "duration of contact" (i.e., the time during which a point remained at rest immediately preceding a given moment t) is characterized by the condition $t(0) = t$, where $t_\infty = \lim_{t\to\infty} t$. Equation (1.21) represents an equation with aftereffect, where the functional is defined on the solutions. The following types of solutions correspond to the physical meaning of the system:

1. **Equilibria at** $|F| \leq f_0 > 0$: These form a "stagnation zone" where f_0 represents the threshold.
2. **Equilibria at** $v = 0$ **and** $f_0 < |F| < f_\infty$: For these states, the velocity remains zero.
3. **Motions** $x(t)$ **passing through points where** $|F| > f_\infty$ **and** $v = 0$: For these cases, the duration of contact t becomes zero, and on some interval (t, τ), the velocity $v(t) \neq 0$.

independently

$y(t) \to \infty$. For motions passing through the points $y = 0$, where $\dot{y} \neq 0$ on some interval (t_1, t_2), it follows that $k = 0$. Consequently, there exists an interval (t_1, τ) on which $y(t) \neq 0$, since $\dot{y} = -ky + \ldots$. As is easily seen, all motions approach the "stagnation zone" and transition into the equilibrium states y^0; the case $y \to \infty$ is impossible. Taking this into account, along with the form of the function Φ, the function f can be represented as follows:

$$f = \Phi(D^{-1}y(t)) = |y|^k \text{sgn } y, \quad k > 0$$

Equation (1.21) can be rewritten in the form of (1.19):

$$\Phi(y, \dot{y}, D^{-1}y) = \begin{cases} y = 0, & D^{-1}y = 0 \\ y = 0, & D^{-1}y \neq 0 \end{cases} \tag{1.22}$$

$$-y - f_d \text{sgn } y \quad \text{for } y > 0,$$

$$0 \quad \text{for } y = 0, |y| < f_d,$$

$$\Phi(y, \dot{y}, D^{-1}y) = \begin{cases} y = 0, & |y| < f_d, D^{-1}y \neq 0 \\ -y + f_d & \text{for } y = 0, f_d < |y| \end{cases}$$

$-y + fA \operatorname{sgn} y$ for $y = 0$, $|f| < |A|$.

All equilibria and motions of types 1° through 4° are strong right-sided solutions to problem (1.22), and these exhaust the set of such solutions. Weak right-sided solutions to problem (1.22) for $f \neq 0$, in addition to types 2° and 4°, include, for example, the following: $y(t) = \pm t^2$. At $t = 0$, this can be interpreted as motion arising from equilibrium in accordance with N. N. Krasovskii' s hypothesis ([**?**], p. 95), which states that "a trajectory point in the phase space of a system cannot remain indefinitely long

on the surfaces of discontinuity" of the right-hand sides of the equations, because it is knocked "into the region of continuity···by random forces and displacements." For $t > 0$, where $y(t) \neq 0$, $y(0) = 0$, and $\dot{y}(0) = 0$, these do not satisfy the original equation (1.21) over the entire interval—where t is the positive root of the equation—and therefore cannot be its right-sided solutions. Thus, problems (1.21) and (1.22) are not equivalent with respect to weak right-sided solutions. We also note that A. F. Filippov' s solutions and the OIV-solutions of problem (1.21) do not encompass the equilibria.

References

Vorlesungen fiber reelle Fiunktionen. Leipzig, Japanese Journ.

Math., 215–230, Heidelberg Akad. Wiss., Math.-naturw. Klasse, Abhandl., 1929.

Persidskii, K. P. *On the Theory of Stability of Solutions of Differential Equations.* Doctoral Dissertation, Vol. 5, 6, 1946.

Viktorovskii, E. E. *Matem. Sb.* (76), 1954, p.

6. Pontryagin

Bibliography

1. Pontryagin, L. S., and Boltyanskii, V. G. *Proceedings of the All-Union Mathematical Congress*, 1956.
2. Andre, J., and Seibert, P. Collection of translations. *Mechanics*, 5 (45), 1957.
3. Andre, J., and Seibert, P. *Proceedings of the International Congress of the International Federation of Automatic Control (IFAC).* USSR Academy of Sciences, 1961.

4. Aizerman, M. A., and Gantmacher, F. R. *Proceedings of the International Congress of the International Federation of Automatic Control (IFAC).* USSR Academy of Sciences, Vol. 5, pp. 659-669, 1961.

5. Krasovskii, N. N. *Certain Problems in the Theory of Stability of Motion.* Fizmatgiz, Moscow, 1959.

6. Filippov, A. F. *Matematicheskii Sbornik*, 51 (93), No. 1, pp. 99-128, 1960.

7. Filippov, A. F. *Proceedings of the International Congress of the International Federation of Automatic Control (IFAC).* USSR Academy of Sciences, 1961.

Bull. Math. France, 1-^{38}, Compositio Math., 89—127, S. K.

Bull. Sci. Math., 139-160; *Differential Equations and Their Applications*, Prague, 229-242; Alimov, Yu. I., *Avtomatika i Telemekhanika*, 22, No. 6; Barbashin, E. A., and Alimov, Yu. I., *Doklady Akademii Nauk SSSR*, 140, No. 1, 1961; Bourbaki, N., *Functions of a Real Variable*, Nauka; Bohl, P. G., "On differential inequalities," *Selected Works*, Izd. AN LatvSSR, Riga; Myshkis, A. D., *Uchenye Zapiski Kharkovskogo Universiteta*, 138, *Zapiski Mekhaniko-Matematicheskogo Fakulteta i Kharkovskogo Matematicheskogo Obshchestva*, 30, 152-163; Mukharlyamov, R. G., *Differentsial' nye Uravneniya*, 3, No. 3, 387-392, 1967; Ishlinsky, A. Yu., *Mechanics of Gyroscopic Systems*, Izd. AN SSSR; Andronov, A. A., Vitt, A. A., and Khaikin, S. E., *Theory of Oscillations*, Moscow, Gostekhizdat; Kurzweil, J., *Czechoslovak Mathematical Journal*, 7, (82), No. 3, 418-449, 1957; Milman, V. D., and Myshkis, A. D., *Siberian Mathematical Journal*, 1, No. 2, 233-244, 1960.

Ann. Soc. Pol. Math., 126-133; Bokshtein, M. F., *Uchenye Zapiski Moskovskogo Universiteta, Matematika*, 15, 3-72; Persidskii, K. P., *Izvestiya AN KazSSR, Seriya Fiziko-Matematicheskikh Nauk, Matematika i Mekhanika*, No. 1, 17-24. Received by the editors of the Kazan Aviation Institute on June 7, 1966. *) Although there is a peculiar robustness of the equilibria 2° for $|x| < f$ in the sense of their invariance under small additional active forces, these equilibria are Lyapunov unstable, which may consequently initiate motion.

Note: Figure translations are in progress. See original paper for figures.

Source: RussiaRxiv —Machine translation. Verify with original.