# CRITERIA FOR DISSIPATIVITY AND THE EXISTENCE OF PERIODIC SOLUTIONS OF PULSED AUTOMATIC SYSTEMS WITH ONE NONLINEAR BLOCK

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## Abstract

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# CRITERIA FOR DISSIPATIVITY AND THE EXISTENCE OF PERIODIC SOLUTIONS OF PULSED AUTOMATIC SYSTEMS WITH ONE NONLINEAR BLOCK

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For a broad class of pulsed systems with a finite number $v$ of degrees of freedom and one nonlinear block, the forced processes occurring in these systems can be described by difference equations of the form

$$x_{\tau+1} = f(\tau, x_\tau), \qquad \tau = 0, \pm 1, \pm 2, \dots,$$

$$f(\tau, x) = Px + q\varphi(\sigma) + k(\tau, x), \qquad \sigma = r^* x. \tag{1}$$

Here $x, q, r$ are real $v \times 1$-vectors (the asterisk denotes transposition); $P$ is a real $v \times v$-matrix whose spectrum lies inside the unit circle; $\varphi(\sigma)$ is a scalar real continuous function of $\sigma$; $k(\tau, x)$ is a real $v \times 1$-vector function, continuous in $x$ uniformly with respect to $\tau$, given for all $\tau$, $\tau = 0, \pm 1, \pm 2, \dots$. It is assumed that the nonlinearity $\varphi(\sigma)$ and the function $k(\tau, x)$, which characterizes the external influences, satisfy the conditions

$$\text{a)} \quad \lim_{|\sigma| \to \infty} \frac{\varphi(\sigma)}{\sigma} \left[ a_0 - \frac{\varphi(\sigma)}{\sigma} \right] \geq 0, \qquad \text{b)} \quad \beta_1 \leq \frac{\varphi(\sigma_2) - \varphi(\sigma_1)}{\sigma_2 - \sigma_1} \leq \beta_2 \tag{2}$$

for arbitrary values $\sigma_1, \sigma_2$, where $\beta_1 \leq 0$, $\beta_2 \geq a_0 > 0$ are some prescribed numbers, $\beta_1 \neq -\infty$, $\beta_2 \neq +\infty$;

$$\lim_{|x| \to \infty} |x|^{-1} k(\tau, x) = 0 \qquad \text{uniformly in } \tau. \tag{3}$$

From the frequency characteristic $\chi(e^{i\omega}) = r^*(P - e^{i\omega} I)^{-1} q$ of the linear part of system (1) and the numbers $a_0, \beta_1, \beta_2$ introduced in (2), define the functions

$$\Phi_0(\lambda) = a_0^{-1} + \operatorname{Re}\chi(\lambda);$$

$$\Pi_\vartheta(\lambda) = (-1)^\vartheta \left[\operatorname{Re}(1-\lambda)\chi(\lambda) + \frac{1}{2}\beta_\vartheta |(1-\lambda)\chi(\lambda)|^2\right], \qquad \vartheta = 1, 2;$$

$$\theta(\lambda) = |1-\lambda|^2 \left[1 + (\beta_1 + \beta_2)\operatorname{Re}\chi(\lambda) + \beta_1\beta_2|\chi(\lambda)|^2\right];$$

$$\Phi_\vartheta(\lambda, \eta, \xi) = \Phi_0(\lambda) - \eta\Pi_\vartheta(\lambda) + \xi\theta(\lambda), \tag{4}$$

where $\eta, \xi$ are some real numbers.

**Theorem 1.** Let $\det \|q \vdots Pq \vdots \cdots \vdots P^{\upsilon-1}q\| \neq 0$. Define the number $\vartheta$ ($\vartheta = 1$ or 2) from the condition

$$(-1)^\vartheta \lim_{|\sigma|\to\infty} \sigma^{-2} \left[\int_0^\sigma \varphi(\sigma')\, d\sigma' - \frac{1}{2}\sigma\varphi(\sigma)\right] \geq 0. \tag{5}$$

Suppose that there exist numbers $\eta \geq 0$ and $\xi \geq 0$ such that $\Phi_\vartheta(e^{i\omega}, \eta, \xi) > 0$ for all $\omega \in [0, \pi]$. Then system (1) has at least one solution bounded for all $\tau$, $-\infty < \tau < +\infty$, and is dissipative (i.e., in the space $\{x\}$ there exists a bounded closed domain $F$ such that: 1) every solution $x_\tau$ of system (1), starting from

of some number $\tau = \gamma$, enters this domain, and 2) from $x_\gamma \in F$ it follows that $x_\tau \in F$ for all $\tau > \gamma$).

**Theorem 2.** For $\xi = 0$, the assertion of Theorem 1 is valid for any $\vartheta = 1, 2$ without assumption (5), and also in the case when $\det \|q_{iPq}j\| \ldots \|P^{\nu-1}q\| = 0$.

**Theorem 3.** Suppose that in equations (1) the nonlinearity $\varphi(\sigma)$ is everywhere differentiable, and $k(\tau, x)$ is a $\nu \times 1$ vector-function periodic in $\tau$ with integer period $\varkappa$. Assume that there exists a number $\eta \geq 0$ for which $\Phi_\vartheta(e^{i\omega}, \eta, 0) > 0$ for all $\omega$, $0 \leq \omega \leq \pi$ (see (4)), where either a) $a_0 = \beta_2$, $\vartheta = 1$*, or b) in the case $\beta_1 = 0$ (see (2)), $\vartheta = 2$ (no additional conditions are imposed on $a_0$). Then system (1) has at least one solution $x_\tau$, $\varkappa$-periodic for all $\tau$, $-\infty < \tau < +\infty$.

Analogously ($^{1,2}$) it is easy to show that Theorems 1-3 are also valid in the critical case (when some of the eigenvalues of the matrix $P$ are located on the unit circle), if the inequalities $\Phi_\vartheta > 0$ are replaced by $\Phi_\vartheta \geq 0$ and one requires: a)

$$\lim_{|\sigma|\to\infty} [\sigma^{-1}\varphi(\sigma) - \varepsilon]\,[a_0 - \varepsilon - \sigma^{-1}\varphi(\sigma)] \geq 0,$$

where $\varepsilon > 0$ is some fixed (arbitrarily small) number; b) the spectrum of the matrix $P + \alpha qr^*$ is wholly located inside the unit disk for some $\alpha$, $\alpha \in (0, a_0]$. (The conditions on the eigenvalues of the matrix $P$ are then satisfied automatically.)

If, instead of relations (2a), (3), one has $\varphi(0) = 0$, $0 \leq \sigma^{-1}\varphi(\sigma) \leq a_0$ (for all $\sigma \neq 0$), $k(\tau, x) \equiv 0$, then, according to the results of the author' s paper ($^2$), under the conditions of Theorems 1 and 2 the trivial solution $x_\tau \equiv 0$ of system (1) is asymptotically stable in the large in the sense of Lyapunov**.

Finally, we note that Theorems 1-3 are entirely analogous to the corresponding results obtained in the works of V. A. Yakubovich ($^{3,4}$) for the case of continuous-control systems, and supplement the well-known theorem of Ya. Z. Tsypkin ($^5$).

It can be shown that the condition $\Phi_\vartheta(e^{i\omega}, \eta, \xi) > 0$ of Theorem 1 covers no less broad a domain in the space of the parameters of system (1) and the numbers $a_0, \beta_1, \beta_2$ than the corresponding criteria ($^{1,6}$), and a broader one than the condition $\Phi_\vartheta(e^{i\omega}, \eta, 0) > 0$, and still more than $\Phi_\vartheta(e^{i\omega}, 0, 0) > 0$.

The proof of Theorems 1-3 is carried out by the method of matrix inequalities of V. A. Yakubovich (see, for example, ($^{3,4,8}$)), extended to the case of discrete systems, and is based on Lemmas 1, 2 ($^1$), 2, 3 ($^6$), Theorem 3 ($^2$), and the following proposition, which is an analogue of the Nyquist criterion***.

**Lemma.** For a real $\nu \times \nu$ matrix $P$, real $\nu \times 1$ vectors $q, r$, a real variable $\alpha$, $\alpha \in [\alpha_1, \alpha_2]$, and a complex variable $\lambda$, define the $\nu \times \nu$ matrix-function $P(\alpha) = P + \alpha qr^*$ and the scalar function $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$. Assume that all eigenvalues of the matrix $P(\alpha_*)$ are located inside the unit disk for some number $\alpha_*$ satisfying the conditions: a) $\alpha_* = 0$, if $0 \in [\alpha_1, \alpha_2]$; b) $\alpha_*$ is an arbitrary number from the interval $[\alpha_1, \alpha_2]$, if $0 \notin [\alpha_1, \alpha_2]$. Then, in order that the spectra of all matrices $P(\alpha)$, $\alpha_1 \leq \alpha \leq \alpha_2$, be located inside the unit disk****, it is necessary and sufficient that the condition

$$1 + \alpha\chi(e^{i\omega}) \neq 0$$

hold for all $\alpha \in [\alpha_1, \alpha_2]$ and $\omega \in [0, \pi]$.

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* In the sense of conditions (2), the quantity $a_0$ in relation (2a), obviously, may be increased (in particular, to the value $a_0 = \beta_2$). However, in this case the condition $\Phi_\vartheta > 0$, by virtue of (4), becomes more stringent.

** In the paper ($^2$), in the author' s opinion, a condition of the form (5) was omitted and there is an inaccuracy in the corresponding place in the proof of Theorem 1.

*** This proposition was used in papers ($^{1,2,6}$), but was not clearly formulated there. An analogous proposition in a somewhat different (geometric) interpretation was proved in ($^{11}$).

**** This condition on the eigenvalues of the matrices $P(\alpha)$ is equivalent to saying that all linear systems $x_{\tau+1} = P(\alpha)x_\tau$, $\alpha \in [\alpha_1, \alpha_2]$, are asymptotically stable in the sense of Lyapunov.

It can be shown that in the case $a_1 = 0,\ a_2 = a_0$ the last relation always holds if the condition $\Phi_\vartheta > 0$ $(\xi \geq 0,\ \eta \geq 0)$ of Theorems 1-3 is satisfied.

The **proof of the lemma** can be carried out as, in the analogous case of continuous control systems, this was done in ($^7$), or by means of certain techniques of ($^9$).

The **proof of Theorem 1** uses the following technique of ($^{2,3}$). Introduce the notation

$$z = \left\| \begin{array}{c} x \\ \hline \varphi(\sigma) \end{array} \right\|, \qquad Q = \left\| \begin{array}{c|c} P & q \\ \hline 0 & 1 \end{array} \right\|, \qquad d = \left\| \begin{array}{c} 0 \\ \hline 1 \end{array} \right\|, \qquad r_1 = \left| \begin{array}{c} r \\ \hline 0 \end{array} \right|, \qquad n(\tau, z) = \left\| \begin{array}{c} k(\tau, x) \\ \hline 0 \end{array} \right\|,$$

$$r_2 = (Q^* - I)r_1, \qquad \sigma = r^* x = r_1^* z, \qquad \Delta\sigma = r^* f(\tau, x) - r^* x, \tag{6}$$

$$\zeta(\tau, z) = \Delta\varphi = \varphi(\sigma + \Delta\sigma) - \varphi(\sigma), \qquad f_1(\tau, z) = Qz + d\zeta(\tau, z) + n(\tau, z),$$

where the quantities $f, x, \varphi, P, q, r, k$ are the same as in equations (1). It is easy to verify that equations (1) are equivalent to the system

$$z_{\tau+1} = f_1(\tau, z_\tau), \qquad \tau = 0, \pm 1, \pm 2, \ldots.$$

Consider the Lyapunov function

$$V(x) \equiv V_1(z) = z^* H z + \eta(-1)^\vartheta \int_0^{r_1^* z} \varphi(\sigma)\, d\sigma, \qquad H = H^*, \tag{7}$$

where the $(\nu+1) \times 1$-vectors $z, r_1$ have the form (6), and the real $(\nu+1) \times (\nu+1)$-matrix $H$ and the parameters $\eta, \vartheta$ $(\vartheta = 1$ or $2)$ are to be chosen in accordance with Lemma 1 ($^4$).

According to the above, the discrete "derivative" $\Delta V \equiv V[f(\tau, x)] - V(x)$ of the function (7), taken by virtue of equations (1), can be computed from the formula $\Delta V = V_1[f_1(\tau, z)] - V_1(z)$. Add to the obtained result the identically zero quantity*

$$\Lambda = -z^* R z + 2z^* s\zeta - \xi\zeta^2 + \left[\frac{1}{2}\eta(-1)^\vartheta \beta_\vartheta - \xi\beta_1\beta_2\right] n^* r_1 r_1^* n + z^* \left[\eta(-1)^\vartheta (d r_1^* + \beta_\vartheta r_2 r_1^*) - 2\xi\beta_1\beta_2 r_2 r_1^*\right] n$$

$$+\xi(\beta_1+\beta_2)r_1^*n\zeta-\varphi(\sigma-a_0^{-1}\varphi)-\eta(-1)^\vartheta\left(\varphi\Delta\sigma+\frac{1}{2}\beta_\vartheta\Delta\sigma^2\right)-\xi(\Delta\varphi-\beta_1\Delta\sigma)(\beta_2\Delta\sigma-\Delta\varphi),$$

where

$$R=R^*=a_0^{-1}dd^*-\frac{1}{2}(dr_1^*+r_1d^*)-\frac{1}{2}\eta(-1)^\vartheta(dr_2^*+r_2d^*)+\left[\xi\beta_1\beta_2-\frac{1}{2}\eta(-1)^\vartheta\beta_\vartheta\right]r_2r_2^*,$$

$$s=\frac{1}{2}\xi(\beta_1+\beta_2)r_2,$$

the numbers $a_0,\beta_1,\beta_2$ occur in (2), (4), and the real parameter $\xi$ is to be chosen. With the aid of (6) we find

$$-\Delta V=\Psi(z,\zeta)+\Omega(\tau,x).$$

Here

$$\Psi(z,\zeta)=z^*(G+R)z+2z^*g\zeta+\gamma\zeta^2,\qquad G=H-Q^*HQ,$$

$$-g=Q^*Hd+s,\qquad \gamma=\xi-d^*Hd,$$

and the function $\Omega(\tau,x)$ is easily estimated by means of relations (2), (3) and the inequalities $\eta\geq 0,\ \xi\geq 0$. Namely: for any $\varepsilon>0$ there exist some positive, finite numbers $\omega_0,\omega_1,\xi_0$ such that

$$\Omega\geq-\varepsilon|x|^2-\omega_1|x|-\omega_0$$

for all $x,\ |x|\geq\xi_0$.

By Theorem 3 (2), the conditions of Theorem 1 guarantee the existence of a matrix $H=H^*$ for which $\Psi(z,\zeta)>0$ for all $z,\zeta,\ |z|+|\zeta|\neq 0$.

Substituting the parameters $\eta,\vartheta$ appearing in the conditions of Theorem 1 and the matrix $H$ found above into expression (7), we obtain from the inequality $\Psi>0$ the existence of numbers $\varepsilon>0,\ \varepsilon_1>0$ such that

$$\Psi(z,\zeta)\geq 2\varepsilon|z|^2+\varepsilon_1\zeta^2,$$

and, all the more (see (6)),

$$\Psi(z,\zeta)\geq 2\varepsilon|x|^2.$$

Taking into account the estimate of the function $\Omega$, we obtain

$$-\Delta V \geq \varepsilon|x|^2 - \omega_1|x| - \omega_0$$

for all $x,\ |x| \geq \xi_0$, i.e. the function (7) satisfies conditions a), b) of Lemma 1 ($^1$). Representing, by means of (6), the function (7) in the form

$$V = x^* H_0 x + 2x^* h_0 \varphi + \chi_0 \varphi^2 + + \eta(-1)^{\vartheta} \int_0^{r^* x} \varphi(\sigma)\, d\sigma,$$

where $H_0 = H_0^*$ is a certain $\nu \times \nu$ matrix, $h_0$ is a $\nu \times 1$ vector, and $\chi_0$ is a number; with the aid of the lemma formulated in the present paper and Lemma 3 ($^6$), one can show, analogously to how this was done in ($^6$), that, by virtue of condition (5) of Theorem 1, the relation $\lim_{|x|\to\infty} V(x) = +\infty$ holds. Thus the function (7) satisfies condition c) of Lemma 1 ($^1$). From what has been proved, by Lemma 1 ($^1$), we obtain the assertions of Theorem 1.

The **proof of Theorems 2 and 3** is carried out with the aid of the Lyapunov function

$$V(x) = x^* H x + \eta(-1)^{\vartheta} \int_0^{r^* x} \varphi(\sigma)\, d\sigma, \qquad H = H^*, \tag{8}$$

where the real $\nu \times \nu$ matrix $H$ and the parameters $\eta, \vartheta$ ($\vartheta = 1$ or 2) are to be chosen in accordance with Lemma 1 ($^1$).* Analogously to the preceding discussion, from (1), (8) we find the discrete "derivative" $\Delta V \equiv V[f(\tau, x)] - V(x)$ and add to the resulting expression the quantity

$$-x^* R x + 2x^* s \varphi - \rho \varphi^2 - \varphi(\sigma - \alpha_0^{-1}\varphi) - \eta(-1)^{\vartheta}(\varphi \Delta\sigma + \tfrac{1}{2}\beta_{\vartheta} \Delta\sigma^2) \equiv 0,$$

where

$$R = R^* = -\tfrac{1}{2}\eta(-1)^{\vartheta}\beta_{\vartheta}(I - P^*) r r^* (I - P),$$

$$s = \tfrac{1}{2}[I - \eta(-1)^{\vartheta}(1 + \beta_{\vartheta} r^* q)(I - P^*)] r,$$

$$\rho = \alpha_0^{-1} - \tfrac{1}{2}\eta(-1)^{\vartheta} r^* q (2 + \beta_{\vartheta} r^* q),$$

the quantity $\Delta\sigma$ is defined in (6), and the numbers $\alpha_0, \beta_1, \beta_2$ occur in relations (2), (4). Then $\Delta V$ can be represented in the form

$$-\Delta V = \Psi(x, \varphi) + \Omega(\tau, x),$$

* This expression was obtained in accordance with the $S$-procedure (see ($^{3,7,8}$)), which reduces the problem o

where

$$\Psi(x,\varphi) = x^*(G+R)x + 2x^*g\varphi + \gamma\varphi^2,$$

$$G = H - P^*HP, \qquad -g = P^*Hq + s, \qquad \gamma = \rho - q^*Hq,$$

and for the function $\Omega$ the same estimate is valid as in the case of Theorem 1. With the aid of Lemma 2 ($^6$) one can show that the conditions of Theorems 2 and 3 guarantee the existence of a matrix $H = H^*$ for which $\Psi(x,\varphi) > 0$ for all $x, \varphi$, $|x| + |\varphi| \neq 0$. Therefore, analogously to the case of Theorem 1, the function (8) (with the indicated matrix $H$) satisfies requirements a), b) of Lemma 1 ($^1$). Now, using the lemma formulated in the present paper, Lemma 3 ($^6$), and certain techniques from ($^4$), it is easy to verify that under the conditions of Theorem 2 the function (8) satisfies the requirement $\lim_{|x|\to\infty} V(x) = +\infty$, and under the conditions of Theorem 3, in addition, the inequality $\partial^2 V/\partial x^2 > 0$ for all $x$. Here $\partial^2 V/\partial x^2$ is the $\nu \times \nu$ matrix of second derivatives of $V$ with respect to the components of the vector $x$. From what has been proved, by Lemmas 1, 2 ($^1$), as well as in ($^1$), the assertions of Theorems 2, 3 follow.

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* A frequency condition for the absolute stability of equations of the form (1) with the aid of a function analogous to (8) was first obtained by Szegö ($^{10}$) for the case when, instead of conditions (2a), (3), the following are satisfied:

$$\varphi(0) = 0, \quad 0 < \sigma^{-1}\varphi(\sigma) \le \alpha_0 \quad \text{for } \sigma \ne 0, \quad k(\tau, x) \equiv 0.$$

*Note: Figure translations are in progress. See original paper for figures.*

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