

# IMPROVING THE SMOOTHNESS OF FUNCTIONS IN A. N. KOLMOGOROV' S THEOREM ON SUPERPOSITIONS

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**Abstract**

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*MATHEMATICS*

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## IMPROVING THE SMOOTHNESS OF FUNCTIONS IN A. N. KOLMOGOROV'S THEOREM ON SUPERPOSITIONS

*(Presented by Academician A. N. Kolmogorov on 20 II 1967)*

By A. N. Kolmogorov's theorem <sup>(1)</sup>, every continuous function of  $n$  variables can be represented in the form

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} \chi_i \left( \sum_{j=1}^n \varphi_{i,j}(x_j) \right), \quad (\text{A})$$

where all the functions are continuous, and the  $\varphi_{i,j}(x_j)$  are fixed and do not depend on the function being decomposed  $f(x_1, x_2, \dots, x_n)$ . G. M. Henkin showed that the  $\varphi_{i,j}(x_j)$  can be chosen to satisfy a Hölder condition with any exponent  $0 < \alpha < 1$ . A. G. Vitushkin and G. M. Henkin proved that if continuously differentiable functions are fixed as the  $\varphi_{i,j}(x_j)$ , then the set of superpositions of the form (A) is nowhere dense in the space of all continuous functions <sup>(2,3)</sup>. Here we shall show that the inner functions in A. N. Kolmogorov's construction can be chosen to satisfy a Lipschitz condition. The precise formulation of the result is as follows.

**Theorem.** *Every function continuous on the cube  $0 \leq x_j \leq 1$  ( $j = 1, 2, \dots, n$ ) can be represented in the form*

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} \chi_i \left( \sum_{j=1}^n \varphi_{i,j}(x_j) \right),$$

where  $\chi_i(x)$  are continuous functions, and  $\varphi_{i,j}(x_j)$  are preassigned fixed functions, independent of the function  $f(x_1, x_2, \dots, x_n)$ , and satisfying a Lipschitz condition.

For simplicity of exposition we shall give the proof for  $n = 2$ . The proof will use the following lemma, whose proof may be found in <sup>(1)</sup>.

**Lemma 1.** (*A. N. Kolmogorov*). Let  $P_{\alpha,j}^k$  ( $k = 1, 2, \dots, 5$ ;  $\alpha = 1, 2, \dots, \alpha_{j,k}$ ;  $j = 1, 2, \dots$ ) be closed rectangles in the square  $I_2$  ( $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ), satisfying the following conditions:

1. For all  $k$  and  $j$ , if  $\alpha_1 \neq \alpha_2$ , then

$$P_{\alpha_1,j}^k \cap P_{\alpha_2,j}^k = \emptyset.$$

2. The diameter of the rectangle  $P_{\alpha,j}^k$  tends to zero as the index  $j$  increases.
3. For every fixed  $j$ , every point of the square  $I_2$  belongs to at least three rectangles from the system  $\{P_{\alpha,j}^k\}$  with one and the same fixed index  $j$ .

Suppose, further, that continuous functions  $\Phi_k(x, y)$  are given such that, for all  $k$  and  $j$ , the sets  $\{\Phi_k(P_{\alpha,j}^k)\}$  ( $\alpha = 1, 2, \dots, \alpha_{j,k}$ ) are pairwise disjoint.

Then, for every continuous function  $f(x, y)$  on  $I_2$ , one can specify continuous functions  $\chi_k(x)$  such that

$$f(x, y) = \sum_{k=1}^5 \chi_k(\Phi_k(x, y)).$$

For the proof of the theorem it suffices to construct a system of rectangles  $P_{\alpha,j}^k$  and functions  $\Phi_k(x, y) = \varphi_k(x) + \psi_k(y)$  satisfying the Lipschitz conditions and Lemma 1.

**Construction of the systems of rectangles.** On the segment  $I_1$  ( $0 \leq x \leq 1$ ) we construct a system  $\{S_{i,j}^k\}$  of intervals  $S_{i,j}^k$  ( $k = 1, 2, \dots, 5$ ;  $i = 1, 2, \dots, i_{j,k}$ ;  $j = 1, 2, \dots$ ) of length  $\Delta_{i,j}^k$ , possessing the following properties:

- a) for any  $k$  and  $j$ , if  $i_1 \neq i_2$ , then  $S_{i_1,j}^k \cap S_{i_2,j}^k = \emptyset$ ;
- b)  $\Delta_{i,j}^k \rightarrow 0$  as  $j \rightarrow \infty$ ;
- c) for every  $j$ , each point of  $I_1$  belongs to at least 4 intervals of the system  $\{S_{i,j}^k\}$  with one and the same fixed  $j$ .

The construction of  $S_{i,j}^k$  will be carried out by induction on  $j$ . For  $j = 1$  we take  $i_{1,k} = 1$  and put  $S_{1,1}^k = I_1$ . We shall assume that the intervals  $S_{i,n-1}^k$  have been constructed. From the set  $\{S_{i,n-1}^k\}$  choose an interval  $S_{i_1,n-1}^{k_1}$  such that

$$\Delta_{i_1,n-1}^{k_1} \geq \Delta_{i,n-1}^k$$

for all  $i$  and  $k$ . Near its center, for example at a distance not exceeding  $\frac{1}{4}\Delta_{i_1,n-1}^{k_1}$ , fix a point  $x_{i_1,n-1}^{k_1}$  which is not an endpoint of the intervals  $\{S_{i,n-1}^k\}$ . By property c) this point belongs to either 4 or 5 intervals of the system  $\{S_{i,n-1}^k\}$ . Having fixed  $x_{i_1,n-1}^{k_1}$ , we arrive at one of the following two cases.

1.  $x_{i_1, n-1}^{k_1}$  belongs to 5 intervals  $\{S_{i, n-1}^k\}$ . Then a sufficiently small interval  $\lambda \subset S_{i_1, n-1}^{k_1}$  with  $\lambda \ni x_{i_1, n-1}^{k_1}$  will be covered by these same intervals. Remove from  $S_{i_1, n-1}^{k_1}$  a sufficiently small interval belonging to  $\lambda$ , containing the point  $x_{i_1, n-1}^{k_1}$ , and lying at a positive distance from the endpoints of the intervals of the system  $\{S_{i, n-1}^k\}$ . We obtain two intervals  $L$  and  $R$  (left and right). In this case, for  $k \neq k_1$  put  $S_{i, n}^k = S_{i, n-1}^k$ . As  $\{S_{i, n}^{k_1}\}$  take the system consisting of the intervals  $L$  and  $R$  and of the intervals  $\{S_{i, n-1}^{k_1}\}$ , with the exception of the interval  $S_{i_1, n-1}^{k_1}$ .
2.  $x_{i_1, n-1}^{k_1}$  belongs to 4 intervals  $\{S_{i, n-1}^k\}$ . This means that for some  $p$ ,

$$x_{i_1, n-1}^{k_1} \notin \bigcup_i S_{i, n-1}^p.$$

In this case remove from the segment  $S_{i_1, n-1}^{k_1}$  an interval  $\lambda \ni x_{i_1, n-1}^{k_1}$  of such small length that it is possible to indicate an interval  $B_{i_1, n-1}^p$  such that

$$B_{i_1, n-1}^p \cap \bigcup_i S_{i, n-1}^p = \emptyset$$

and  $B_{i_1, n-1}^p \supset \lambda$ . Let us note here that the length  $\Delta_n$  of the interval  $B_{i_1, n-1}^p$  can be chosen arbitrarily small.  $\Delta_n$  will be chosen below in the construction of the functions.

In this case, as before, the intervals  $L$  and  $R$  (obtained by removing  $\lambda$  from  $S_{i_1, n-1}^{k_1}$ ) and  $\{S_{i, n-1}^{k_1}\}$  without the interval  $S_{i_1, n-1}^{k_1}$  will constitute  $\{S_{i, n}^{k_1}\}$ . The system  $\{S_{i, n}^p\}$  will consist of the intervals  $\{S_{i, n-1}^p\}$  and the interval  $B_{i_1, n-1}^p$ . For  $k \neq k_1$  and  $k \neq p$  put  $S_{i, n}^k = S_{i, n-1}^k$ .

Thus the systems  $\{S_{i, j}^k\}$  ( $k = 1, 2, \dots, 5$ ) on  $I_1$  have been constructed. (In the assumption that  $\Delta_n$  is given each time when this is necessary. It will be seen below that  $\Delta_n$  is chosen immediately after the  $(n-1)$ -st step of constructing the systems of intervals.) They possess properties a), b), and c).

On the segment  $\bar{I}_1$  ( $0 \leq y \leq 1$ ), in an analogous way, we construct systems of intervals  $\{T_{i, j}^k\}$ . (In constructing the systems  $\{T_{i, n}^k\}$ , in analogy with  $L, R, B_{i_1, n-1}^p$ , and  $x_{i_1, n-1}^{k_1}$ , there arise the intervals  $\bar{L}, \bar{R}, \bar{B}_{i_1, n-1}^p$  and the point  $y_{i_1, n-1}^{k_1}$ .)

Denote

$$P_{(i, l), j}^k = \{(x, y) : x \in S_{i, j}^k, y \in T_{l, j}^k\}.$$

Renumber the pairs  $(i, l)$  by the index  $\alpha$  and put

$$P_{\alpha, j}^k = P_{(i(\alpha), l(\alpha)), j}^k.$$

The system of rectangles  $\{P_{\alpha, j}^k\}$  satisfies the conditions of Lemma 1.

**Construction of the functions.**  $\varphi_k(x)$  and  $\psi_k(y)$  will be obtained as the limits of certain sequences  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  as  $j \rightarrow \infty$ .

We shall construct  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  so that they satisfy the following conditions:

) for fixed  $j$  and  $k$ , the functions  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  are constant on the intervals  $S_{i,j}^k$  and  $T_{i,j}^k$ , respectively, and linear in the intervals complementary to  $\bigcup_i S_{i,j}^k$  and  $\bigcup_i T_{i,j}^k$ , respectively;

)  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  are nondecreasing functions;

) the function

$$\Phi_{k,j}(x, y) = \varphi_{k,j}(x) + \psi_{k,j}(y)$$

is such that, for any fixed  $k$  and  $\tau \leq j$ , and for  $a_1 \neq a_2$ ,

$$\rho(\Phi_{k,j}(P_{a_1,\tau}^k), \Phi_{k,j}(P_{a_2,\tau}^k)) > \varepsilon_\tau > 0,$$

where  $\Phi_{k,j}(P_{a,\tau}^k)$  is the set of values of the function  $\Phi_{k,j}(x, y)$  on the rectangle  $P_{a,\tau}^k$ , and  $\rho(B, C)$  is the distance between the sets  $B$  and  $C$ ;

) the functions  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  satisfy a Lipschitz condition with constant  $< 1$ ;

)

$$\max_x |\varphi_{k,j-1}(x) - \varphi_{k,j}(x)| \leq 1/2^j, \quad \max_y |\psi_{k,j-1}(y) - \psi_{k,j}(y)| \leq 1/2^j$$

(condition ( )) is needed only to ensure the uniform convergence of the functions  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  as  $j \rightarrow \infty$ ).

The functions  $\varphi_{k,j}(x)$  and  $\psi_{k,j}(y)$  are constructed by induction.

For  $j = 1$ ,  $k = 1, 2, \dots, 5$ , put  $\varphi_{k,1}(x) = \psi_{k,1}(y) \equiv 0$ . The passage from the functions  $\varphi_{k,n-1}(x)$  and  $\psi_{k,n-1}(y)$  to the functions  $\varphi_{k,n}(x)$  and  $\psi_{k,n}(y)$  is connected with the construction of the intervals  $\{S_{i,n}^k\}$  and  $\{T_{i,n}^k\}$ , and is therefore carried out by one of the following two methods.

I. In case 1 (see the construction of the systems of intervals), for  $k \neq k_1$  we take

$$\varphi_{k,n}(x) = \varphi_{k,n-1}(x)$$

and

$$\psi_{k,n}(y) = \psi_{k,n-1}(y).$$

For  $k = k_1$ , on all intervals  $S_{i,n}^{k_1}$ , except  $\text{and}$ , put  $\varphi_{k_1,n}(x) = \varphi_{k_1,n-1}(x)$ , and on  $T_{i,n}^{k_1}$ , except  $\text{and}$ , put  $\psi_{k_1,n}(y) = \psi_{k_1,n-1}(y)$ .

Put

$$\varphi_{k_1,n}() = \varphi_{k_1,n-1}() + \varepsilon_{n,1}, \quad \psi_{k_1,n}() = \psi_{k_1,n-1}() - \varepsilon_{n,2},$$

$$\varphi_{k_1,n}() = \varphi_{k_1,n-1}() + \varepsilon_{n,3}, \quad \psi_{k_1,n}() = \psi_{k_1,n-1}() + \varepsilon_{n,4},$$

and in the intervals complementary to  $\bigcup_i S_{i,n}^{k_1}$  and  $\bigcup_i T_{i,n}^{k_1}$ , respectively, define  $\varphi_{k_1,n}(x)$  and  $\psi_{k_1,n}(y)$  to be linear and to agree at the endpoints of the intervals with the already assigned values.

Choose the numbers  $\varepsilon_{n,\beta} > 0$  so that  $\varphi_{k_1,n}(x)$  and  $\psi_{k_1,n}(y)$  still satisfy conditions ( ), ( ), ( ), and ( ). (We note that under condition ( ), for sufficiently small  $\varepsilon_{n,\beta}$ , it is necessary to monitor only the case  $\tau = n$ , since for small changes of  $\varphi_{k_1,n-1}(x)$  and  $\psi_{k_1,n-1}(y)$  the obtained functions  $\varphi_{k_1,n}(x)$  and  $\psi_{k_1,n}(y)$  satisfy condition ( ) for  $\tau < n$ .)

II. In case 2 (see the construction of the systems of intervals), for  $k \neq k_1$  and  $k \neq p$  put

$$\varphi_{k,n}(x) = \varphi_{k,n-1}(x)$$

and

$$\psi_{k,n}(y) = \psi_{k,n-1}(y).$$

On the intervals

$$S_{i,n}^p \neq B_{i_1,n-1}^p$$

and

$$T_{i,n}^p \neq \overline{B}_{i_1,n-1}^p$$

respectively, take

$$\varphi_{p,n}(x) = \varphi_{p,n-1}(x)$$

and

$$\psi_{p,n}(y) = \psi_{p,n-1}(y).$$

Put

$$\varphi_{p,n}(B_{i_1,n-1}^p) = \varphi_{p,n-1}(x_{i_1,n-1}^{k_1}) + \varepsilon'_n, \quad \psi_{p,n}(\overline{B}_{i_1,n-1}^p) = \psi_{p,n-1}(y_{i_1,n-1}^{k_1}) + \varepsilon''_n,$$

where  $\varepsilon'_n$  and  $\varepsilon''_n$  will be chosen below.

In the intervals complementary to  $\bigcup_i S_{i,n}^p$  and  $\bigcup_i T_{i,n}^p$ , respectively, define  $\varphi_{p,n}(x)$  and  $\psi_{p,n}(y)$  to be linear and to agree at the endpoints of the intervals with the already assigned values. Recall that we could choose the length  $\Delta_n$  of the intervals  $B_{i_1,n-1}^p$  and  $\overline{B}_{i_1,n-1}^p$  arbitrarily. We now choose it so that, for sufficiently small  $\varepsilon'_n$  and  $\varepsilon''_n$ ,  $\varphi_{p,n}(x)$  and  $\psi_{p,n}(y)$  satisfy condition ( ). Now choose  $\varepsilon'_n$  and  $\varepsilon''_n$  so that the functions  $\varphi_{p,n}(x)$  and  $\psi_{p,n}(y)$  satisfy conditions ( ), ( ), ( ), and ( ).

The functions  $\varphi_{k_1,n}(x)$  and  $\psi_{k_1,n}(y)$  are obtained from the functions  $\varphi_{k_1,n-1}(x)$  and  $\psi_{k_1,n-1}(y)$  in the same way as in case 1.

Now set

$$\varphi_k(x) = \lim_{j \rightarrow \infty} \varphi_{k,j}(x), \quad \psi_k(y) = \lim_{j \rightarrow \infty} \psi_{k,j}(y),$$

$$\Phi_k(x, y) = \varphi_k(x) + \psi_k(y).$$

Since for  $\Phi_{k,j}(x, y)$  condition (e) is fulfilled for every  $j$ , the  $\Phi_k(x, y)$  satisfy the conditions of Lemma 1. From condition ( ) it follows that  $\varphi_k(x)$  and  $\psi_k(y)$ , for every  $k$ , satisfy a Lipschitz condition with constant 1. The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*

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