ON THE PERFECT IMAGE OF A PARACOMPACT FEATHERED SPACE

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.83

MATHEMATICS

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ON THE PERFECT IMAGE OF A PARACOMPACT FEATHERED SPACE

(Presented by Academician P. S. Aleksandrov, December 2, 1966)

The definitions, needed for understanding what follows, of a perfect mapping, a support, a star, and a feathered space (= p-space), as well as information on feathered spaces, may be found in ([2]). The main result of the present paper is Theorem 1, which is a solution of a problem posed by A. V. Arhangel' skii in 1963 in ([2]) and then repeated in ([1])*.

Theorem 1. *A perfect image of a paracompact p-space** is a paracompact p-space***.*

Proof. The paracompactness of the image follows from Michael' s theorem ([4]): the closed image of a paracompact space is paracompact (for a proof see, for example, ([1])).

The proofs of Lemmas 1, 2, 3, and 5 below are obvious.

Lemma 1. *Let λ be any system of sets in an arbitrary space X. The set of points at which λ is locally finite is open in X.*

Lemma 2. *Let $\{\gamma_\alpha\}_{\alpha \in A}$ be a certain set of families of subsets of an arbitrary space X, and let $\gamma_\alpha x$ denote the star of the point x with respect to the family γ_α. Then*

$$x \in \bigcap_{\alpha \in A} \gamma_\alpha y \Longleftrightarrow y \in \bigcap_{\alpha \in A} \gamma_\alpha x.$$

Lemma 3. *Let γ be an arbitrary family of subsets of an arbitrary space X, and let*

$$y \in \bigcap_{\gamma \ni U \ni x} U.$$

Then $\gamma y \supseteq \gamma x$.

Lemma 4 (basic). *Let X be paracompact, and let λ be an arbitrary family of sets open in βX and covering X.*

Then into λ one can inscribe a family λ' of sets open in βX, covering X, and satisfying the conditions:

1) *for any $x \in \beta X$, $[\lambda' x] \subseteq U \in \lambda$;*

2) *λ' is locally finite at the points of the set $\bigcup\{\Gamma : \Gamma \in \lambda'\}$.*

Proof of Lemma 4. Denote by γ the covering of the space X formed by those of its open subsets whose closures in βX are contained in elements of λ. Inscribe in γ a locally finite covering γ_1 (by open subsets of X) such that, for every point $x \in X$, $\gamma_1 x$ is contained in some element of λ. To each element $V \in \gamma_1$ assign an open set $U = U(V)$ in βX satisfying the condition $U \cap X = V$. Obviously, when $[U]_{\beta X}$ is contained in

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* The work was carried out under the supervision of A. V. Arhangel' skii in the topological seminar of P. S. Aleksandrov at Moscow University.

** All spaces considered below are assumed in advance to be completely regular T_1-spaces.

*** Theorem 1 is equivalent (see ([2])) to the following theorem:

Theorem 1′. *Let $f : X_1 \to Y$ and $g : X_1 \to X_2$ be perfect mappings, with X_2 a metrizable space; then Y can be mapped perfectly onto some metrizable space.*

some element λ. Put $\lambda_1 = \{U(V) : V \in \gamma_1\}$. The family λ_1 is locally finite at all points of the set X: indeed, let $x \in X$, let O'_x be a neighborhood of x in X meeting only a finite set of elements of the cover γ_1, and let O_x be some open set in βX such that $O_x \cap X = O'_x$; then O_x meets precisely those elements of the family λ_1 whose traces* on X meet O'_x, by the density of X in βX. The set G of points of the space βX at which λ_1 is locally finite is open in βX by Lemma 1 and, as we have just proved, contains X. Put $\lambda' = \{\Gamma = U \cap G : U \in \lambda_1\}$. The family λ' is locally finite at all points of the set $\bigcup\{\Gamma : \Gamma \in \lambda'\}$, covers X, and the family of closures of its elements is inscribed in λ. Consequently, Lemma 4 will be proved if we verify that, whatever the point $x \in \beta X$, $[\lambda' x] \subseteq O \in \lambda$. But the set $\lambda'_x = \{\Gamma : \Gamma \in \lambda',\ \Gamma \ni x\}$ is empty or finite by what was said above. The first case is trivial; consider the second. The set $\bigcap\{\Gamma : \Gamma \in \lambda'_x\}$ is then nonempty (it contains x) and open in βX. From the density of X in βX it follows that there exists

$$x' \in X \cap \left(\bigcap\{\Gamma : \Gamma \in \lambda'_x\}\right).$$

By Lemma 3, $\lambda' x' \supseteq \lambda' x$. But

$$\lambda' x' \subseteq [\lambda_1 x']_{\beta X} \subseteq [\gamma_1 x']_{\beta X},$$

by the construction of λ_1 and γ_1, and moreover

$$\gamma_1 x' \subseteq A \subseteq [A]_{\beta X} \subseteq O,$$

where $A \in \gamma$, $O \in \lambda$. Thus we have

$$[\lambda' x]_{\beta X} \subseteq [\lambda' x']_{\beta X} \subseteq [\gamma_1 x']_{\beta X} \subseteq [A]_{\beta X} \subseteq O \in \lambda,$$

as required.

Lemma 5. Let X, Y be bicompacta, $X \subseteq Y$, $X = \bigcap G_n$, the G_n open in Y, $[G_{n+1}] \subseteq G_n$, and $X_x \in G_n$.

Then every neighborhood of the bicompactum X contains all points x_n, beginning with some one.

We now proceed to the proof of Theorem 1. After Lemma 4 we may assume that the elements of a feathering $\{\gamma_n\}$ of X in βX are covers of the set X by sets open in βX, satisfying two conditions:

1. Each cover is locally finite at the points covered by it.
2. The star of each point with respect to the succeeding cover is contained in some element of the preceding cover together with its closure.

I. Let $x \in \beta X \setminus X$. Then

$$\bigcap_n \gamma_n x \subseteq \beta.$$

This follows from Lemma 2 and the definition of a feathered space.

II. $B \subseteq X$ is a bicompactum $\Rightarrow \bigcap_n \gamma_n B \subseteq X$.

Let $x \in \beta X \setminus X$ and $x \in \bigcap_n \gamma_n B$. This means that in every γ_n there is $U_n \in \gamma_n$, $x \in U_n$, $U_n \cap B \neq \emptyset$. But $[\gamma_n x] \subseteq \gamma_{n-1} x$, and therefore

$$B_n = B \cap [\gamma_n]$$

is a decreasing sequence of nonempty bicompacta; their intersection is nonempty and lies in X, which contradicts I.

III. $B \subseteq X$ is a bicompactum $\Rightarrow [\gamma_n B] \subseteq \gamma_{n-1} B$.

First of all, the number of elements of γ_n meeting B is finite,—this is an obvious consequence of the local finiteness of γ_n and the bicompactness of B. The closure of the star of the bicompactum B with respect to γ_n is equal to the union of the closures of the sets comprising it. For each of these sets we find an element of γ_{n-1} that contains it together with its closure. The union of the latter will contain $[\gamma_n B]$ and be contained in $\gamma_{n-1} B$.

IV. Put

$$\sigma_n = \{\gamma_n f^{-1} y : y \in Y\},$$

where f is the perfect mapping of the space X onto the space Y under consideration. Let $y \in Y$ and $B = f^{-1} y$; then

$$\bigcap_n \sigma_n B \subseteq X.$$

If there exists $x \in \beta X \setminus X$ and $x \in \bigcap_n \sigma_n B$, then for every n there is

$$B_n = f^{-1} y_n, \quad y_n \in Y,$$

such that

$$\gamma_n B_n \cap B \neq \emptyset$$

and

$$\gamma_n B_n \ni x.$$

Equivalent relations:

$$\gamma_n B \cap B_n \neq \emptyset$$

and

$$\gamma_n x \cap B_n \neq \emptyset.$$

Choose arbitrarily

$$a_n \in \gamma_n B \cap B_n$$

and

$$b_n \in \gamma_n x \cap B_n.$$

The sets

$$\bigcap_n \gamma_n B = \bigcap_n [\gamma_n B]$$

and

$$\bigcap_n \gamma_n x = \bigcap_n [\gamma_n x]$$

are bicompacta.

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* The trace of U on X is $U \cap X$.

By virtue of II, the first of them lies in X; by virtue of I, the second lies in $\beta X \setminus X$; consequently, their images under $F : \beta X \to \beta Y$ do not intersect.* On the other hand, we are in the hypotheses of Lemma 5. Consider the closed sets $F\left(\bigcap_n \gamma_n B\right) \subseteq Y$ and $F\left(\bigcap_n \gamma_n x\right) \subseteq \beta Y \setminus Y$ and their disjoint neighborhoods U and V. The sets $F^{-1}(U)$ and $F^{-1}(V)$, by Lemma 5, contain a_n and b_n, respectively, starting with some n. But then U and V contain all points $f(B_n)$, starting with the same index, i.e. $U \cap V \neq \emptyset$, which contradicts the choice of U and V. Thus,

$$\bigcap_n \sigma_n B \subseteq X.$$

V. We now construct a feathering $\{\delta_n\}$ of the space Y in βY: as an element of the cover δ_n we declare the complement of the image of the complement of any element of the cover σ_n (i.e.

$$\delta_n = \{\, U = \beta Y \setminus F(\beta X \setminus V) : \ V \in \sigma_n \,\}, \quad n = 1, 2, \ldots).$$

Let us show that $\{\delta_n\}$ indeed constitutes a feathering. First make sure that δ_n covers Y. Consider an arbitrary point $y \in Y$. We have $\gamma_n f^{-1} y \in \sigma_n$, $y \in \beta Y \setminus F(X \setminus \gamma_n f^{-1} y) \in \delta_n$. Let us now show that

$$\bigcap_n \delta_n y \subseteq Y$$

for every $y \in Y$. Indeed,

$$F^{-1}\left(\bigcap_n \delta_n y\right) \subseteq \bigcap_n F^{-1} \delta_n y \subseteq \bigcap_n \sigma_n f^{-1} y$$

(since $F^{-1}\delta_n$ is, obviously, inscribed in σ_n)

$$\subseteq X$$

(by what was proved above). But

$$F^{-1}(\beta Y \setminus Y) = \beta X \setminus X.$$

Hence,

$$\bigcap_n \delta_n y \subseteq Y.$$

Theorem 1 is proved.

Let us give some consequences of the theorem just proved, concerning the notion of coabsoluteness (see ([5])) and strengthening the corresponding results of V. I. Ponomarev from ([5]). Their basis is the following Theorem 2, which is evident after Theorem 1.

Theorem 2. *The class of paracompact feathered spaces is closed under multi-valued perfect mappings.*

Theorem 3. *A space coabsolute with a paracompact feathered space is itself a paracompact feathered space.*

Theorem 10.2 from ([5]) gives us the following criterion.

Theorem 4. *A space is coabsolute with a metrizable one if and only if it admits a single-valued perfect irreducible mapping onto some metrizable space.*

Theorem 10.1 from ([5]) can now be formulated as follows.

Theorem 5. *In order that a space be coabsolute with a metrizable one, it is necessary and sufficient that it be a paracompact p-space and that one of the two following conditions be fulfilled in it:*

1. *There exists in it a dense σ-locally finite system of open sets.*
2. *There exists in it a dense σ-discrete system of open sets.*

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Received
23 XI 1966

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* A perfect mapping $f : X \to Y$ is uniquely extendable to a mapping $F : \beta X \to \beta Y$, and $F(\beta X \setminus X) = \beta Y \setminus Y$ (see ([3])).

Note: Figure translations are in progress. See original paper for figures.