ON THE WELL-POSEDNESS AND PSEUDO-WELL-POSEDNESS OF PROBLEMS

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Abstract

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ON THE WELL-POSEDNESS AND PSEUDO-WELL-POSEDNESS OF PROBLEMS

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1. Let a mapping A be given from a topological space R_1 into a topological space R_2: $y = Ax$, x ranges over P_A, y ranges over Q_A ($P_A \subset R_1$, $Q_A \subset R_2$). Let the equation

$$Ax = g, \qquad g \in R_2 \tag{1}$$

be considered with respect to x; g is an arbitrary fixed element.

As usual ([1,2]), we shall say that problem (1) is **well-posed** or **correctly formulated** if the following conditions are satisfied:

I. The equality $Q_A = R_2$ holds and the space R_2 is complete.
II. The mapping A is one-to-one.
III. The mapping A^{-1} is continuous.

If at least one of the listed conditions is violated, we shall say that problem (1) is **ill-posed** or **incorrectly formulated**.

2. Let a translation T be given from a topological space R_1 into a topological space R_2: $\eta = Tx$, x ranges over P_T, and Q_T is the union of the sets η. (It is assumed that the topological spaces occurring in the article are once and for all equipped with neighborhoods and generalized neighborhoods; moreover, in metric spaces, spheres of points are taken as neighborhoods, and spheres of sets as generalized neighborhoods.)

We shall say that the translation T is **partially continuous of the i-th type** ($i = 0, 1, 2, 3, 4, 5, 6, 7$) at $x = x_0$, if $x_0 \in P_T$ and there exists a sequence $W_T^{(m)}(x_0)$, $m = 1, 2, ...$, of subsets of the set P_T such that $x_0 \in W_T^{(m)}(x_0)$, $\overline{P}_T = \bigcup_m W_T^{(m)}(x_0)$, and for every m the translation $\eta = Tx$, x ranges over $W^{(m)}(x_0)$, $\eta \subset Q_T$, is continuous of the i-th type at the point x_0.

We shall say that the translation T is **partially continuous of the i-th type** ($i = 0, 1, 2, 3, 4, 5, 6, 7$), if it is partially continuous at every point of the set P_T and if, for every point $x_0 \in P_T$ and every sequence of points x_k, $k = 1, 2, ...$, of the set P_T converging in R_1 to x_0, the sequence of sets $W_T^{(m)}(x_k)$, $k = 1, 2, ...$, converges in R_1 of the seventh type to the set $W_T^{(m)}(x_0)$ $(m = 1, 2, ...)$ ([3]).

It is easy to see that the concept of partial continuity of the i-th type generalizes the concept of continuity in various directions. Thus, if the translation T is partially continuous of the i-th type at the point x_0 and $W_T^{(m)}(x_0) = P_T$, then the translation T is continuous of the i-th type at this point. If, however, the translation T is partially continuous of the i-th type and one-to-one, then it represents a partially continuous mapping.

We shall say that the translation T is **strictly partially discontinuous of the i-th type** if it is partially continuous of the i-th type, but is not continuous of the i-th type $(i = 0, 1, 2, 3, 4, 5, 6, 7)$.

3. In the case where problem (1) is ill-posed, let us introduce the following notions.

A translation $\xi = Bg$, where g ranges over R_2, $\xi \subset R_1$, will be called a **generalized solution** of problem (1) if $Bg = A^{-1}g$ when g ranges over Q_A. A generalized solution will be called **complete** if the translation B is complete.

A translation $\xi = B^*g$, where g ranges over R_2, $\xi \subset R_1$, will be called a **partial generalized solution** of problem (1) if $B^*g \subset A^{-1}g$ when g ranges over Q_A.

A mapping $x = B_1 g$, where g ranges over R_2, $x \in R_1$, will be called a **partial generalized solution** of problem (1) if $B_1 g \subset A^{-1}g$ when g ranges over Q_A.

We shall call problem (1) **pseudo-well-posed**, or **pseudo-correctly posed**, if the following conditions are satisfied:

A. The space R_2 is complete.
B. There exists at least one generalized solution $\xi = Bg$, where g ranges over R_2, $\xi \subset R_1$, of problem (1).
C. The translation B is partially continuous of zero type.

We shall call problem (1) **partially pseudo-well-posed** if, in the conditions of the preceding definition, the translation B constitutes a partial generalized solution of problem (1).

We shall call problem (1) **partially well-posed** if the following conditions are satisfied:

A′. The space R_2 is complete.
B′. There exists at least one partial generalized solution $x = B_1 g$, where g ranges over R_2, $x \in R_1$, of problem (1).
C′. The mapping B_1 is partially continuous.

It is clear that the notions of pseudo-well-posedness, partial pseudo-well-posedness, and partial well-posedness generalize, in different directions, the notion of well-posedness. Thus every well-posed problem is pseudo-well-posed, partially pseudo-well-posed, and partially well-posed.

If, in the conditions of the definition of pseudo-well-posedness (partial pseudo-well-posedness), continuity of the i-th type ($i = 0, 1, 2, 3, 4, 5, 6, 7$) holds, then we shall say that problem (1) is **pseudo-well-posed** (**partially pseudo-well-posed**) of the i-th type.

If, in the conditions of the definition of pseudo-well-posedness (partial pseudo-well-posedness), partial discontinuity of the i-th type ($i = 0, 1, 2, 3, 4, 5, 6, 7$) holds, then we shall say that problem (1) is **pseudo-well-posed** (**partially pseudo-well-posed**) of the $(\alpha + i)$-th type, where α is some transfinite ≥ 0.

If, in the conditions of the definition of pseudo-well-posedness (partial pseudo-well-posedness), purely partial discontinuity of the i-th type ($i = 0, 1, 2, 3, 4, 5, 6, 7$) holds, then we shall say that problem (1) is **pseudo-well-posed** (**partially pseudo-well-posed**) purely of the $(\alpha + i)$-th type, in order to emphasize that $\alpha > 0$.

If, in the conditions of the definition of pseudo-well-posedness, the translation B is single-valued, we shall say that problem (1) is **pseudo-well-posed with uniqueness**.

Theorem 1. *If problem (1) is pseudo-well-posed of the* $(\alpha + 7)$*-th type, then the translation* A^{-1} *admits a unique continuous extension of the* $(\alpha + 7)$*-th type to the set* $\overline{Q}_A$.

It follows from the theorem that if the set Q_A is everywhere dense in R_2 and problem (1) is pseudo-well-posed of the $(\alpha + 7)$-th type with respect to some complete generalized solution, then the latter is determined in a unique way.

A generalized solution of problem (1) will be called **minimal** if it constitutes a minimal extension of the translation A^{-1}.*

* An extension $\tilde{B}$ of a translation B to the set $P_{\tilde{B}}$ is called **minimal** if the translation $\tilde{B}$ on the set $P_{\tilde{B}} \setminus P_B$ is single-valued.

Theorem 2. If problem (1) is pseudowell-posed of type $(a + i)$ ($i = 0, 1, 2, 3, 4, 5, 6, 7$) with respect to some minimal generalized solution, then the translation A^{-1} admits a unique minimal discontinuous continuation of type $(a + i)$ to the set $\overline{Q}_A$.

It follows from the last theorem that if the set Q_A is everywhere dense in R_2 and problem (1) is pseudowell-posed of type $(a + i)$ ($i = 0, 1, 2, 3, 4, 5, 6, 7$) with respect to some minimal generalized solution, then the latter is determined uniquely.

4. Let us dwell on some examples.

Example 1. Suppose two real Euclidean spaces are given: R_1 of dimension n and R_2 of dimension m. Let a real matrix $A = [a_{ik}]$ of size $(m \times n)$, of rank r, be given. Consider the equation

$$Ax = g, \qquad g \in R_2 \tag{2}$$

with respect to x; g is an arbitrary given element. If $r = n = m$, then problem (2), as is well known, is well-posed. In the other cases it is, for one reason or another, ill-posed. We define a generalized solution of problem (2) as follows. Consider the functional

$$\Phi_g x = \|Ax - g\|^2, \qquad x \text{ ranges over } P_A \tag{3}$$

and denote by Bg the set of elements minimizing it. This set is always nonempty ([3]). Since for every $g \in Q_A$ the equality $Bg = A^{-1}g$ holds, the translation B represents a generalized solution of problem (3).

Theorem 3. If $r = n < m$, then problem (3) is pseudowell-posed with uniqueness, and the mapping B is continuous.

Theorem 4*. If $r < n \leq m$ or $r \leq m < n$, then problem (3) is pseudowell-posed of the 7th type.

Example 2. Let H be a real Hilbert space and let A be a distributive operator (a mapping from H into H) with real eigenvalues and with a system of orthonormal eigenvectors complete in the subspace P_A. Consider the equation

$$Ax = g, \qquad g \in H \tag{4}$$

with respect to x; g is an arbitrary fixed element.

It is easy to see that problem (4) is ill-posed except in the case when A is a topological mapping of the space H onto itself.

To define a generalized solution of (4), consider the functional

$$\Phi_g x = \|\tilde{A}x - g\|^2, \qquad x \text{ ranges over } P_{\tilde{A}}, \tag{5}$$

where $\tilde{A}$ is the minimal distributive continuation of the operator A with closed range. We denote by Bg the set of elements minimizing the functional (5) ([4]). Since for every $g \in Q_A$, $Bg = A^{-1}g$, B constitutes a generalized solution of problem (4).

Theorem 5. If zero is neither an eigenvalue of the operator A nor a limit point of its eigenvalues, then problem (4) is pseudowell-posed with uniqueness, and the mapping B is continuous.

Theorem 6. If zero, being a limit point of the eigenvalues of the operator A, is not an eigenvalue of it, then problem (4) is pseudowell-posed with uniqueness, and the mapping B is purely partially discontinuous.

Theorem 7. If zero, being an eigenvalue of the operator A, is not a limit point of its eigenvalues, then problem (4) is pseudowell-posed of the seventh type.

Theorem 8. If zero is both an eigenvalue of the operator A and a limit point of its eigenvalues, then problem (4) is pseudowell-posed of pure type $(a + 7)$.

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* The condition of Theorem 4 should be read as follows: if the mapping B of the space R_1 onto R_2 is open and the translation B^{-1} is finite-valued.

Note: Figure translations are in progress. See original paper for figures.