On a class of motions of a pendulum in a medium with large resistance

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Abstract

Full Text

ON A CLASS OF PENDULUM MOTIONS IN A MEDIUM WITH HIGH RESISTANCE

In this paper, we consider the motion of a pendulum in a medium where the resistance is proportional to the velocity. The dynamics are governed by a second-order differential equation, and we focus on a specific class of motions characterized by high resistance coefficients.

1. Problem Statement

The equation of motion for a pendulum in a resisting medium can be expressed in the following form:

$$\ddot{\theta} + k\dot{\theta} + \omega^2 \sin\theta = 0$$

where θ represents the angular displacement, k is the resistance coefficient, and ω^2 is related to the gravitational acceleration and the length of the pendulum. We are particularly interested in the behavior of the system when k is sufficiently large, leading to what is often termed the "overdamped" regime or motion in a highly viscous medium.

[Figure 1: see original paper]

2. Analysis of the Motion

When the resistance coefficient k is large, the qualitative behavior of the solutions to (1) changes significantly compared to the undamped case. By applying the transformation of variables and analyzing the phase plane, we can establish the existence of specific trajectories.

Let us consider the characteristic equation associated with the linearized version of (1):

$$\lambda^2 + k\lambda + \omega^2 = 0$$

The roots of this equation are given by:

$$\lambda_{1,2} = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$$

For the case of high resistance, specifically when $k^2 > 4\omega^2$, the roots λ_1 and λ_2 are real and negative. This implies that the system does not exhibit oscillatory behavior but instead returns to the equilibrium position asymptotically.

3. Asymptotic Behavior

We investigate the asymptotic properties of the solutions as $t \to \infty$. In the regime of high resistance, the motion is dominated by the exponential decay terms. The general solution for the linearized system can be written as:

$$\theta(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are constants determined by the initial conditions $\theta(0)$ and $\dot{\theta}(0)$.

As shown in , the rate of convergence to the equilibrium point $\theta = 0$ is heavily dependent on the magnitude of k.

1. Consider the differential equation

$$\ddot{x} + a\dot{x} + f(x) = 0, \quad (1)$$

where a is a positive constant and $f(x)$ is a continuous, piecewise differentiable periodic function with period 2π. It is assumed that the following relation holds:

$$\Delta = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx < 0 \quad (2)$$

along with the conditions:

$$f(0) = f(x_1) = 0, \quad f'(0) > 0, \quad f'(x_1) < 0, \quad (3)$$

where $x = 0$ and $x = x_1$ $(0 < x_1 < 2\pi)$ are the only zeros of the function $f(x)$ on the set $[0, 2\pi)$. Denoting $x_2 = 2\pi$, we also obtain the inequality $f'(x_2) > 0$. A schematic representation of the graph of the function $f(x)$ satisfying conditions (2) and (3) is shown in [Figure 1: see original paper]. A particular case of equation (1) is the equation describing the oscillations of a pendulum. Equation (1) is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -f(x) - ay \quad (4)$$

which, in turn, is equivalent to the equation

$$\frac{dy}{dx} = \frac{-ay - f(x)}{y} \quad (5)$$

The properties of the solutions to equation (1) have been investigated in several works [?, ?, ?, ?, ?]. The primary result of these studies can be summarized as follows. There exists a critical value $a = a_*$ such that for $a < a_*$, equation (5) possesses a periodic solution $y = Y(x)$. This solution corresponds to a limit cycle of system (4) that encircles the phase cylinder. As time increases, all other trajectories asymptotically approach either this limit cycle or one of the equilibrium positions. In the case where $a > a_*$, system (4) has no periodic trajectories; every trajectory asymptotically approaches one of the system' s equilibrium positions as time increases.

In the case $a = a_*$, the system possesses a trajectory connecting its unstable saddle points $(\pm 2k\pi, 0)$, where $k = 0, \pm 1, \pm 2, \ldots$. The solution $y = Y_*(x)$ of equation (5) corresponding to this trajectory is also periodic. However, the motion of the point in phase space corresponding to this solution is not periodic, as it corresponds to a trajectory that approaches unstable singular points as $t \to \infty$ and $t \to -\infty$. We note that the periodic solutions $y = Y(x)$ and $y = Y_*(x)$ of equation (5) are positive-definite solutions; specifically, the inequalities $Y(x) > 0$ and $Y_*(x) > 0$ hold for all $-\infty < x < \infty$. Equation (5) does not possess any negative-definite periodic solutions [?].

Below, we demonstrate that for certain parameter values, equation (5) may possess a sign-alternating periodic solution $Z(x)$. This solution corresponds to an integral curve passing through the stable and unstable singular points of system (4). The corresponding motion of the point in phase space is an aperiodic movement from an unstable equilibrium position to a stable one [Figure 1: see original paper]. From the arguments presented below, it follows that there exists a bifurcation value of the parameter $a = a_{**}$ such that for $a = a_{**}$, equation (5) has a sign-alternating periodic solution $y = Z(x)$, while for $a < a_{**}$, this solution does not exist. We also prove that under certain additional constraints on the function $f(x)$, the inequality $a_{**} < \infty$ holds. For a very broad class of functions $f(x)$, an upper bound for the bifurcation value a_{**} of equation (5) is provided. It is interesting to note that for the classical pendulum equation

$$\ddot{x} + a\dot{x} + b \sin x = 0$$

The bifurcation value is calculated precisely as $2/\pi$. The trajectories adjacent to the singular points O_j of system (4) are referred to as separatrices and are denoted as shown in [Figure 2: see original paper]. The angular coefficients of the tangents to the separatrices at point O_j are determined from the following equation:

$$k^2 + ak + f'(x_i) = 0 \quad (6)$$

It is evident that an alternating periodic solution to equation (5) exists if the point $(0, 0)$ is a singular point of the "node" type and if the separatrices Γ converge to this point. Thus, a necessary condition for the existence of an alternating periodic solution is the satisfaction of the inequality

$$a^2 > 4f'(0) \quad (7)$$

It is known [?] that the inequality $r(0) < s(0)$ holds, where $r(0)$ and $s(0)$ denote the ordinates of the intersection points of the separatrices Γ with the axis, respectively. Conversely, if the inequality is satisfied, we have $r(0) > s(0)$. Since $r(0) = 0$ when $a > a_{cr}$, and given that $s(0) > 0$ always holds, it follows that this inequality is also a necessary condition for the existence of an alternating periodic solution to equation (5). Therefore, if the value of a_{cr} is finite, the following inequality must necessarily be satisfied:

$$a > \max(2\sqrt{f'(0)}, a_{cr}) \quad (8)$$

We specifically note the following fact: equation (5) cannot possess any alternating periodic solutions other than the specified solution, the trajectory of which consists of the separatrices Γ meeting at the point $(0, 0)$. Indeed, the line $y = 0$ is an isocline of vertical slopes. If an integral curve $y = Z(x)$ intersects the x-axis at an ordinary point $x = x_0$, then the abscissa of the representative point must reach an extremal value at $x = x_0$, beyond which the function cannot be extended. This is because for $y > 0$ we have $y' > 0$, and for $y < 0$ we have $y' < 0$. Consequently, the curve $y = Z(x)$ can only intersect the x-axis at singular points. However, since $s(0) > 0$ always holds, such a curve can only be composed of the separatrices Γ, which proves our assertion.

We now provide an example demonstrating that equation (5) can indeed possess an alternating solution. Let us define the function $f(x)$ on the interval $-\pi < x < \pi$ as follows:

$$f(x) = -k(x - a) \text{ for } -a < x \leq a \quad (9)$$

At all other points, we extend the function f periodically. Let us further assume that the parameter value satisfies the condition $a > 2\sqrt{k}$. In this case, the origin is a singular nodal point for system (4), through which two integral lines pass. The remaining trajectories near the origin approach the point $(0, 0)$, being tangent to the first of the aforementioned integral lines. Through the saddle point $(-a, 0)$ passes an integral line which coincides with the separatrix. Since the separatrix enters a region bounded by the straight line and the separatrix $y = n(x)$, from which it clearly cannot exit, this separatrix must inevitably approach the origin as $x \to \infty$ [Figure 3: see original paper]. Due to the oddness of the function, the separatrix of the system under consideration also approaches the origin. Thus, in the case being considered, we have the following theorem.

Theorem 1. Let the following conditions be satisfied:

$$|f(x)| < |h(x)|, \quad \text{sign } h(x) = \text{sign } f(x) \quad (10)$$

where $f(x)$ is a continuous function. If the equation possesses an alternating periodic solution $y = Z(x)$, then for the equation $h(x)$, there also exists an alternating periodic solution satisfying the inequality $|h(x)| < |Z(x)|$ and the condition $h(x) = Z(x)$ for $x < x_0$. Indeed, if the equation has an alternating periodic solution, it follows from the comparison principle [**?**] that on a given interval, the separatrix corresponding to equation (11) will be located below the curve, while on another interval, the separatrix of the equation will lie above this curve. Thus, by examining the direction field, we see that the separatrices corresponding to equation (11) must meet at the origin to form a periodic integral curve.

It follows from Theorem 1 that there may exist a value such that, for a given function $f(x)$, equation (5) will possess an alternating periodic solution. By defining a piecewise linear function of the type shown in (9), we can calculate the value for it. This value provides an upper bound for the bifurcation value of the parameter a corresponding to any function $f(x)$ satisfying conditions (10). As an example, let us consider the equation describing the oscillations of a pendulum: $\ddot{x} + a\dot{x} + f(x) = 0$. Since the existence of an alternating periodic solution requires the origin to be a singular point of the "node" type, we obtain the inequality $a > 2\sqrt{g/l}$. On the other hand, since $|\sin x| \leq f(x)$, then by Theorem 1, the equation possesses the required properties.

The equation

$$\ddot{x} + a\dot{x} + b \sin x = 0 \quad (13)$$

possesses an alternating periodic solution. Thus, $a < C \cdot 2\sqrt{b}$ and, consequently, for equation (12) we have $a = 2\sqrt{b}$. The alternating periodic solution of equation (13) corresponds to a monotonic fall of the pendulum from the upper vertical position to the lower vertical position, coming to a complete stop in the latter position. Obviously, such motions are possible only when the viscosity of the medium in which the pendulum oscillates is sufficiently high.

Let there be a given piecewise-linear periodic function $\phi(x)$ (with period 2π), defined as:

$$\phi(x) = \begin{cases} bx & \text{for } -\delta < x < \eta \\ -c(x - x_1) & \text{for } \eta < x < x_1 \end{cases} \quad (14)$$

Lemma. The fulfillment of the relation

$$a^2 > 4b \quad (15)$$

is a necessary and sufficient condition for the existence of an alternating periodic solution $y = Z(x)$ of equation (5) when $f(x) = \phi(x)$; that is, in this case, $a = 2\sqrt{b}$.

The necessity of satisfying relation (15) for the existence of the solution $y = Z(x)$ was indicated above. We shall demonstrate that the presence of a nodal singular point $(0, 0)$ in system (4) when $f(x) = \phi(x)$ is sufficient for the existence of an alternating periodic solution. Indeed, it is easy to see that the integral line corresponding to the separatrix is given by

$$y = \frac{-a + \sqrt{a^2 + 4c}}{2}(x - x_1)$$

Since the origin is a nodal singular point of the system when relation (15) is satisfied, the integral line $m(x)$ passes through the point $(0, 0)$ and is defined by the equation

$$y = \frac{-a - \sqrt{a^2 - 4b}}{2}x = m(x)$$

The lines $y = m(x)$, $x = -\epsilon$, and the x-axis bound a region G. As time increases, the representative point starting from any point within this region moves along a trajectory that does not leave the region and approaches the origin as $t \to \infty$. Comparing values, we obtain the inequality $Z(-\epsilon) < m(-\epsilon)$, which is equivalent to $a < \sqrt{a^2 + 4c}$. It follows that the representative point, moving along the separatrix $Z(x)$, enters the region G and coincides with the origin as $t \to \infty$ [Figure 1: see original paper]. By analogous calculations, it can be shown that the second separatrix also approaches the origin as $t \to \infty$.

Theorem 2. If the function $f(x)$ is such that the relation $f(x) \le \phi(x)$ holds for all x, where $\phi(x)$ is defined by (14), then equation (5) with $a \ge 2\sqrt{b}$ has an alternating periodic solution; i.e., in this case we have $f'(0) \le a^2/4$.

If we take a relay-type function as the function $f(x)$ in equation (5), defined by:

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$$

we find that the inequality $z(0) > 0$ always holds. The curve cannot cross the line of monotonicity on the interval $(x, 0)$. Thus, $z(0)$ must be positive. In this case, for any value of a, equation (5) does not possess a sign-alternating periodic solution ($a = \infty$).

Theorem 3. If the function $f(x)$ is continuously differentiable and $f''(0)$ exists and is finite, then the bifurcation value a is also finite.

Indeed, $f(x) = f'(0)x + ...$ Let us consider the curve

$$y = kx + Ax^2 \quad (17)$$

where k is a root of the equation

$$k^2 + ak + f'(0) = 0 \quad (18)$$

We choose A such that the integral curves of system (4) intersect curve (17) in a decreasing direction. This is ensured if the total derivative along the trajectories is negative. Using (16) and (17), we obtain

$$\dot{W} = -[(k^2 + ak + f'(0))x + (3k + a)Ax^2 + \dots]$$

From condition (18), it follows that $\dot{W} \approx -(3k+a)Ax^2$. This will be negative if we require $(3k+a)A > 0$. Thus, condition (19) ensures the required positioning of curve (17) on the interval $(-\delta, 0)$. By choosing a large enough, we ensure the separatrix Γ enters the origin.

Similarly, one can choose a large enough such that the separatrix is located above the curve on the interval $(0, \epsilon)$, ensuring its convergence to the origin. Thus, there exists a finite value for which a sign-alternating periodic solution exists. The statement remains valid if we replace the second derivative condition with the Lipschitz-type condition $|f'(x) - f'(0)| < C|x|^\delta$.

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Note: Figure translations are in progress. See original paper for figures.