ON THE SOLVABILITY OF NONLINEAR EQUATIONS

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Abstract

Full Text

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ON THE SOLVABILITY OF NONLINEAR EQUATIONS

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1. In the present note we consider the question of the existence of a solution for the nonlinear equation

$$AF(x) = x \tag{1}$$

in a real Hilbert space. Here A and $F(x)$ are, respectively, linear and nonlinear operators acting in H. Such equations were considered in the works (3,6). It was assumed there that the operator A is self-adjoint and bounded, and that $F(x)$ is a potential operator satisfying certain conditions.

Here we shall consider equation (1) under more general assumptions concerning the operator A; namely, we shall not assume that the operator A is bounded. In Theorem 4 we shall also consider the case when the operator $F(x)$ is not potential. The theorems established here for equation (1) will be applied to the study of nonlinear Hammerstein-type integral equations, and we shall establish new assumptions for them.

2. Theorem 1. *Let A be a positive self-adjoint operator defined on a dense set in the Hilbert space H, and let $f(x)$ be a weakly differentiable and weakly upper semicontinuous functional satisfying the condition*

$$f(x) \leq a_1(x,x)^{\theta} + a_2(x,x)^{\mu} + a_3,$$

where $a_1 < 0,\ a_2 \geq 0,\ a_3 \geq 0,\ 0 \leq \mu < \theta < 1.$
Then the equation $AF(x) = x$, where $F(x) = \operatorname{grad} f(x)$, has at least one solution.

Theorem 2. *Suppose that the following conditions are satisfied:*

1°. *A is a positive self-adjoint operator defined on a dense set in the Hilbert space H; $F(x)$ is a potential operator acting in H.*

2°. *For some fixed element $x_0 \in R(A^{1/2})$ the inequality*

$$(DF(x_0 + \tau h, h), h) \leq \gamma(h)\|h\|, \qquad 0 \leq \tau \leq 1,$$

holds, where $\gamma(h) \leq 0,\ \lim_{\|h\|\to\infty} \gamma(h) = -\infty,\ DF(x,h)$ *is the Gateaux differential of the operator* $F(x)$*.*

3°. *The potential of the operator* $F(x)$ *is weakly upper semicontinuous.*

Then the operator equation $AF(x) = x$ *has at least one solution.*

Remark 1. If, instead of condition 3° of the theorem, one requires that for all $x, h \in H$ the inequality $(DF(x,h),h) \leq 0$ hold, with equality possible only for $h = 0$, then the equation $AF(x) = x$ has a unique solution.

Theorem 3. *Let* A *be a positive self-adjoint operator defined on a dense set in* H*; let* $F(x)$ *be a potential operator satisfying, for some fixed* $x_0 \in R(A^{1/2})$*, the inequality*

$$(F(x_0+h) - F(x_0), h) \leq \gamma(\|h\|)\|h\|,$$

where $\gamma(t) \leq 0,\ \lim_{t\to\infty} \gamma(t) = -\infty$*.*

Then, if the potential of the operator $F(x)$ is weakly upper semicontinuous, the equation $AF(x) = x$ has at least one solution.

Remark 2. If, in the conditions of the theorem, one additionally requires that for any $x, h \in H$ the inequality

$$(F(x+h) - F(x), h) \leq 0$$

hold, with equality only for $h = 0$, then the equation $AF(x) = x$ has a unique solution.

Theorem 4. Let A be a positive self-adjoint operator, defined on a dense set in H; let $F(x)$ be a nonlinear operator satisfying the conditions:

1°.

$$\|F(x+h) - F(x)\| \leq N(r)\|h\|, \quad x,\ x+h \in D_r,$$

where D_r is the ball of radius r with center at the point $x = 0$.

2°.

$$(F(x+h) - F(x), h) \leq m(r)\|h\|^2, \quad x,\ x+h \in D_r,$$

where $m(t) \leq 0$,

$$\lim_{t\to\infty} tm(t) = -\infty.$$

Then the operator equation $AF(x) = x$ has a unique solution.

3. We shall apply the results presented here to establishing existence and uniqueness theorems for the solution of the nonlinear integral equation

$$u(x) = \int_\Omega K(x,y) g(u(y),y)\, dy. \tag{2}$$

Here Ω is a measurable set of finite or infinite measure in an m-dimensional Euclidean space.

Such propositions were considered in the works ([4],[6],[7]). It was assumed there that the integral operator generated by the kernel $K(x, y)$ is self-adjoint and bounded.

However, when kernels of a more general type (Carleman kernels) are considered, this condition may fail. Recall that a Carleman kernel is a function $K(x, y)$, measurable on the set $\Omega \times \Omega$, such that almost everywhere in $\Omega \times \Omega$

$$K(x, y) = K(y, x)$$

and, for almost all $x \in \Omega$,

$$\int_\Omega |K(x, y)|^2 \, dy < \infty.$$

We shall consider equation (2), without assuming boundedness of the operator

$$Au = \int_\Omega K(x, y)u(y) \, dy,$$

generated by the kernel $K(x, y)$.

In what follows we shall require that A be a self-adjoint, positive operator defined on a dense set in H. Examples of kernels generating unbounded integral operators are given in ([1]).

Our basic assumption concerning the nonlinear part $g(u, x)$ consists in requiring continuity of the Nemytskii operator $hu = g(x, u(x))$ from L^p into L^q $(1/p + 1/q = 1)$.

A necessary and sufficient condition for continuity of hu was established in ([2]).

The following propositions hold:

1°. Let $p = 2$, and let the functional

$$f(u) = \int_\Omega G(u(x), x) \, dx$$

be the potential of the operator hu.

Suppose further that

$$G(u, x) \leq a_1 u^2 + a_2(x)|u|^\alpha + a_3(x),$$

where $a_1 < 0,\ 0 \leq a_2(x) \in L^{2/(2-\alpha)},\ 0 < \alpha < 2$ and $0 \leq a_3(x) \in L^1$.

Then, if $f(u)$ is weakly upper semicontinuous, equation (2) has at least one solution belonging to $L^2(\Omega)$.

2°. Let the function $g(u, x)$ satisfy the following conditions:

1) $g(u,x) \leq a_1 u + a_2(x)|u|^\alpha + a_3(x)$, $u \geq 0$;

2) $g(u,x) \geq b_1 u + b_2(x)|u|^\alpha + b_3(x)$, $u < 0$;

here $a_1 < 0$, $b_1 < 0$; $0 \leq a_2(x) \in L^{2/(1-\alpha)}$; $0 \geq b_2(x) \in L^{2/(1-\alpha)}$; $0 < \alpha < 1$; $0 \leq a_3(x) \in L^2$; $0 \geq b_3(x) \in L^2$*.

Then, if the potential of the operator hu is weakly upper semicontinuous, equation (2) has at least one solution belonging to $L^2(\Omega)$.

The proof of these propositions is based on Theorem 1.

3°. Let H be the function $g(u,x)$ such that for all u, $v \in (-\infty, +\infty)$ and almost all $x \in \Omega$

$$(g(u+v,x) - g(u,x))v \leq av^2,$$

where $a < 0$**.

Then equation (2) has a unique solution in the space $L^2(\Omega)$.

In the next proposition we shall assume that the measure of the set Ω is infinite.

4°. Let A be a positive self-adjoint operator (bounded or unbounded) acting in the space $L^2(\Omega)$. Suppose, moreover, that A is a bounded operator from L^p into L^q

$$(1/p + 1/q = 1).$$

Assume further that H is the function $g(u,x)$ satisfying the following conditions:

1) $g(u,x) \leq a_2(x)|u|^\alpha + a_3(x), \quad u \geq 0,$

$$g(u,x) \geq -a_2(x)|u|^\alpha - a_3(x), \quad u < 0;$$

here $0 \leq a_2(x) \in L^{p/[(p-1)-\alpha]}$, $0 < \alpha < 1$, $0 \leq a_3(x) \in L^q$.

2) The Nemytskii operator $hu = g(u(x), x)$ is continuous from L^p into L^q.

Then, if the functional $f(u)$, which is the potential of the operator hu, is weakly upper semicontinuous, the integral equation (2) has at least one solution belonging to the space $L^p(\Omega)$.

Let us note that analogous propositions can also be established for systems of nonlinear integral equations of the form

$$u_i(x) = \int_\Omega K_i(x,y) g_i(u_1(y), u_2(y), \dots, u_n(y), y)\, dy,$$

$i = 1, 2, \dots, n$. In this case we additionally assume the existence of a function $G(u_1, u_2, \dots, u_n, x)$ such that

$$g_i(u_1, u_2, \dots, u_n, x) = \frac{\partial}{\partial u_i} G(u_1, u_2, \dots, u_n, x).$$

The proof of these propositions is based on the method proposed by us in (7).

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REFERENCES

[1] N. I. Akhiezer, UMN, 2, no. 5 (21), 93 (1947).
[2] M. M. Vainberg, DAN, 92, no. 2, 213 (1953).
[3] M. M. Vainberg, *Variational methods for the study of nonlinear equations*, 1956.
[4] M. M. Vainberg, Uchen. zap. Mosk. obl. ped. inst., 77, no. 5, 131 (1959).
[5] M. M. Vainberg, Sibirsk. matem. zhurn., 2, 2, 201 (1961).
[6] M. Golomb, Math. Zs., 39, 45 (1934).
[7] M. M. Lavrent' ev, DAN, 166, no. 2, 284 (1966).
[8] Ya. L. Engel' son, Nauchn. dokl. vyssh. shkoly, no. 4, 75 (1958).

* The existence of such constants $a_1,\ b_1,\ \alpha$ and functions $a_i(x),\ b_i(x)$, $i = 1, 2$, follows from the continuity condition for the Nemytskii operator.

** This condition, in particular, will be fulfilled if $g(u, x)$ has a partial derivative with respect to u, $g'_u(u, x)$, which for almost all $x \in \Omega$ satisfies the inequality $g'_u(u, x) \leq a$ $(a < 0)$ and is bounded below.

Note: Figure translations are in progress. See original paper for figures.