PROPERTIES OF MAPPINGS OF CERTAIN CHEBYSHEV SYSTEMS

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Abstract

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PROPERTIES OF MAPPINGS OF CERTAIN CHEBYSHEV SYSTEMS

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Let $\mathcal{M}$ denote an $(n+1)$-dimensional subspace of the space $C[a, b]$ of continuous real functions on $[a, b]$.

Definition 1. $\mathcal{M}$ has the **Chebyshev property** if it has a basis forming a Chebyshev system (see ([1,2])).

Definition 2. $\mathcal{M}$ has **bounded oscillation** if every function $w \in \mathcal{M}$ ($w \neq$ const) has $\leq n+1$ local extrema, and is strictly monotone between them (two local extrema are always present at the points a and b).

Definition 3. $\mathcal{M}$ is called **suitable** if, for any numbers $\beta_0, \beta_1, \dots, \beta_n$ such that $(\beta_j - \beta_{j-1})(\beta_{j+1} - \beta_j) < 0$, there exist $w \in \mathcal{M}$ and points $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ such that $w(t_j) = \beta_j$, with w strictly monotone on each interval (t_{j-1}, t_j).

Definition 4. $\mathcal{M}$ is called **strongly suitable** if: 1) it contains the constants and 2) for every k ($k = 1, \dots, n$) it is suitable, i.e., for any numbers $\beta_0, \beta_1, \dots, \beta_k$ such that $(\beta_j - \beta_{j-1})(\beta_{j+1} - \beta_j) < 0$, there exist $w \in \mathcal{M}$ and points $a = t_0 < t_1 < \dots < t_k = b$ such that $w(t_j) = \beta_j$, with w strictly monotone on each interval (t_{j-1}, t_j).

Definition 5. $\mathcal{M}$ is called **weakly suitable** if, for any numbers $\beta_0, \beta_1, \dots, \beta_n$ such that $(\beta_j - \beta_{j-1})(\beta_{j+1} - \beta_j) < 0$, there exist $w \in \mathcal{M}$ and points $a = t_0 < \dots < t_n = b$ such that

$$w(t_j) = \beta_j = \underset{\min}{\max}\{w(t) : t \in [t_{j-1}, t_{j+1}]\} \qquad (j = 1, \dots, n-1).$$

Proposition 1. *Every $(n+1)$-dimensional subspace contains at least one element that has $\geq n$ zeros and at least one element that has $\geq n+1$ local extrema.*

Proposition 2. *If $\mathcal{M}$ has bounded oscillation, then it has the Chebyshev property, but the converse is false.*

Proposition 3. *Every suitable subspace is weakly suitable. Every strongly suitable subspace is suitable.*

The properties described are topological: if ψ is a homeomorphism of some interval onto $[a, b]$, and $\mathcal{M}\psi$ is the subspace of functions $v \circ \psi$ ($v \in \mathcal{M}$, $v \circ \psi(t) = v[\psi(t)]$), then $\mathcal{M}\psi$ has any one of the properties indicated in Definitions 1-5 if and only if $\mathcal{M}$ has the same property.

The suitability property of a subspace is of interest because the choice of the β_j in accordance with Definitions 3 or 4 (but not 5) is equivalent to specifying the tree of the function under consideration in the sense of A. S. Kronrod (^{3}); specifying such a tree is equivalent to choosing the topological type of functions in the above sense. If the tree of a function of one variable is finite, then it is a sequence of edges alternately going upward and downward. Thus, if, say, $\mathcal{M}$ is strongly suitable, this means that every function $v \in C[a, b]$ whose tree has $\leq n$

of edges, topologically equivalent to some $w \in \mathcal{M}$, i.e., $v = w \circ \psi$, where ψ is some orientation-preserving homeomorphism of the interval onto itself.

The considerations given partially explain why the following theorem has attracted the attention of various authors.

Theorem A. *The space $\mathcal{P}_n$ of algebraic polynomials of degree $\leq n$ is strongly suitable.*

A. J. Kemper (^{4,5}) proved a weaker assertion than Theorem A: that every $v \in \mathcal{C}[a, b]$ whose tree has $\leq n$ edges is weakly topologically equivalent to some $p \in \mathcal{P}_n$, i.e., that $v = \varphi \circ p \circ \psi$, where ψ and φ are some orientation-preserving homeomorphisms of the interval $[a, b]$ onto itself.

Theorem A was first proved by the author (^{6}). A weaker theorem was proved independently and by a very similar method by J. Mycielski and S. Paszkowski (^{7}). J. Kammerer (^{8}) gave another proof of Theorem A, applying ideas from approximation theory. R. Thom* (1961) indicated a proof of Theorem A based on embedding the space of real polynomials into the space of complex polynomials. He used as a lemma the following complex variant of Theorem A, which was independently established by J. W. Andrushkiw (^{9}):

Theorem B. *For any complex numbers $\beta_1, \dots, \beta_{n-1}$ there exist complex numbers $z_1, \dots, z_{n-1}$ and a polynomial p of degree n such that $p(z_j) = \beta_j$, $p'(z_j) = 0$ $(j = 1, \dots, n-1)$.*

I. J. Schoenberg noted that Theorem A can be derived from the work of G. R. MacLane (^{10}).

Recently, Theorem A was generalized to broader classes of Chebyshev subspaces than $\mathcal{P}_n$. R. S. Johnson (^{11}) gave a proof for monosplines (of a certain class of piecewise-polynomial functions). V. S. Videnskii (^{12}) and, somewhat later, the author (^{13}), independently established very similar theorems. Then V. S. Videnskii (^{14}) strengthened the result:

Theorem C. *Every Chebyshev subspace $\mathcal{M}$ is weakly suitable. Moreover, for given $\beta_0, \dots, \beta_n$ the corresponding function $w \in \mathcal{M}$ is unique.*

Not every Chebyshev subspace is suitable, even for $n = 2$. But the following theorem is true.

Theorem. *Every subspace of bounded oscillation is strongly suitable.*

The proof given below is topological and does not contain a construction of the desired function. As can be shown, the uniqueness property is not preserved, but if the tree of the desired function has n edges, then uniqueness follows from Theorem C.

We begin with a description of the set Δ_m of all trees that have $\leq m$ edges. Until now $\omega \in \Delta_m$ has been determined by specifying $(\beta_0, \dots, \beta_k)$ $(0 < k \leq m)$, with

$$(\beta_j - \beta_{j-1})(\beta_{j+1} - \beta_j) < 0.$$

But instead of this one may specify β_0, the sign of $\beta_1 - \beta_0$ (which henceforth is called the sign of ω), and the numbers

$$\lambda_j = |\beta_j - \beta_{j-1}| > 0 \qquad (j = 1, \dots, k).$$

We shall use the notation

$$\omega = (\beta_0, \pm, \lambda_1, \dots, \lambda_k).$$

A function whose tree is this ω has total variation

$$V = \sum_1^k \lambda_j.$$

For $k = 0$, $\omega = (\beta_0)$ denotes the tree of the function identically equal to β_0. To define a topology in Δ_m, associate with each $\omega \in \Delta_m$ the real function $f_\omega(x)$, whose graph is the broken line with vertices at the points

$$(0, \beta_0),\ (\lambda_1, \beta_1),\ (\lambda_1 + \lambda_2, \beta_2), \dots, (V, \beta_k),$$

and, for $x > V$, put $f_\omega(x) = \beta_k$; for $\omega = (\beta_0)$ put $f_\omega(x) = \beta_0$.

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* This result has not been published.

Then a natural metric is defined on the set of trees:

$$d(\omega_1, \omega_2) = \sup\{|f_{\omega_1}(x) - f_{\omega_2}(x)| : 0 \leq x < \infty\}.$$

Lemma 1. *If $\mathcal{D} \subset \mathcal{C}[a, b]$ is a set such that every $w \in \mathcal{D}$ has a tree with no more than m edges, then the mapping* ind, *which assigns to each $w \in \mathcal{D}$ its tree* ind(w), *is continuous.*

Indeed, let $w_0 \in \mathcal{D}$ and $\varepsilon > 0$. It can be shown that for all $w \in \mathcal{D}$ for which $\|w - w_0\| < \varepsilon/3m$, the inequality

$$d(\operatorname{ind}(w), \operatorname{ind}(w_0)) < \varepsilon$$

holds.

The subsequent proof is based on the study of the mapping ind. First consider the subset K_m of the set Δ_m, consisting of those $\omega = (\beta_0, \pm, \lambda_1, \dots, \lambda_k) \in \Delta_m$ for which $\beta_0 = 0$, $\sum \lambda_j \leq 1$. Its boundary $\partial K_m = \Sigma_{m-1}$ consists of those $\omega \in K_m$ for which $\sum \lambda_j = 1$.

Lemma 2. *K_m is homeomorphic to the m-dimensional ball; Σ_{m-1} is homeomorphic to the $(m-1)$-dimensional sphere.*

We prove the second assertion; the first is not difficult to derive from it. Denote by S^k $(k = 1, \dots, m-1)$ the set of all points in R^m which have the form

$$(\cos\theta_1,\ \sin|\theta_1|\cos\theta_2,\ \sin|\theta_1|\sin|\theta_2|\cos\theta_3, \dots$$

$$\dots,\ \sin|\theta_1|\cdots\sin|\theta_{k-1}|\cos\theta_k,\ \sin|\theta_1|\cdots\sin|\theta_{k-1}|\sin\theta_k)$$

$$(-\pi \leq \theta_j \leq \pi),$$

the last $m-k-1$ coordinates, which are not written, being equal to zero. The set S^k is a k-dimensional sphere, and S^{k-1} is its equatorial $(k-1)$-dimensional sphere.

Define recursively a function F, homeomorphically mapping S^k onto Σ_k, so that the set where $\sin\theta_k > 0$ is carried into the plus half of the set $\Sigma_k \setminus \Sigma_{k-1}$. Let

$$F(1) = (0, +, 1), \qquad F(-1) = (0, -, 1).$$

Suppose that F has already been defined on S^{k-1}; then it remains to define it on the set $S^k \setminus S^{k-1}$, whose points have the form

$$\xi = (\xi_0, \dots, \xi_{k-2}, \xi_{k-1}\cos\theta_k, \xi_{k-1}\sin\theta_k),$$

where

$$(\xi_0, \dots, \xi_{k-1}) \in S^{k-1} \setminus S^{k-2}, \qquad \xi_{k-1} > 0, \qquad \sin\theta_k \neq 0.$$

We use the notation

$$F(\xi_0, \dots, \xi_{k-1}) = (0, +, \lambda_1, \dots, \lambda_k),$$

$$F(\xi_0, \dots, \xi_{k-2}, -\xi_{k-1}) = (0, -, \mu_1, \dots, \mu_k),$$

where $\lambda_j > 0$, $\sum \lambda_j = 1$, $\mu_j > 0$, $\sum \mu_j = 1$.

Put $\gamma = (1 + \cos\theta_k)/2$ and define F on S^k as follows:

$$F(\xi) = (0, +, \gamma\lambda_1, \gamma\lambda_2 + (1-\gamma)\mu_1, \dots, \gamma\lambda_k + (1-\gamma)\mu_{k-1}, (1-\gamma)\mu_k)$$
$$(\sin\theta_k > 0),$$

$$F(\xi) = (0, -, (1-\gamma)\mu_1, \gamma\lambda_1 + (1-\gamma)\mu_2, \dots, \gamma\lambda_{k-1} + (1-\gamma)\mu_k, \gamma\lambda_k)$$
$$(\sin\theta_k < 0).$$

It is not hard to verify that the mapping thus constructed has the required properties.

Corollary. Δ_m *is homeomorphic to the Euclidean space* R^{m+1}.

Proof of the theorem. Let $\mathcal{M}$ have a bounded variation. Consider the set $\mathcal{B}$ of functions $w \in \mathcal{M}$ such that $w(a) = 0$ and whose total variation is ≤ 1. Since $\mathcal{B}$ is a closed convex body in R^n, it is homeomorphic to the n-dimensional ball. The boundary $\partial\mathcal{B}$ consists of the functions with total variation equal to one. Under the mapping ind, continuous by Lemma 1, $\operatorname{ind}\mathcal{B} \subseteq K_n$, $\operatorname{ind}(\partial\mathcal{B}) \subseteq \Sigma_{n-1} = \partial K_n$. To prove that $\mathcal{M}$ is strongly admissible, it is enough to establish that $\operatorname{ind}\mathcal{B}$ fills all of K_n. If this is not so, then there exists an interior point $w \in K_n$ which is a boundary point for $\operatorname{ind}\mathcal{B}$. The set $\operatorname{ind}\mathcal{B}$ is closed, and the mapping ind cannot be locally one-to-one in a neighborhood of w. But from Lemma 2 we know that K_{n-1} divides K_n into two symmetric cells $K_n^{\pm}$, and it is clear that $\operatorname{ind}\mathcal{B}$ intersects the interior of each cell, so that

it suffices to determine whether a boundary point $\operatorname{ind}\mathfrak{B}$ can be an interior point of $K_n^{\pm}$. The following lemma answers this question.

Lemma 3. *Let* $w \in \mathfrak{B}$ *have* $n+1$ *local extrema. Then, in some neighborhood of* w*, the mapping* ind *is locally one-to-one.*

Indeed, choose $\delta > 0$ so small that $|w(t_j) - w(t_{j-1})| > 2\delta$ for all j, where t_j are consecutive local extrema of the function w; moreover, let w_1 and w_2 be elements of $\mathcal{M}$ whose distance from w is no more than δ. There exist disjoint neighborhoods U_j of the points t_j $(j = 0, 1, \dots, n)$ such that each of the functions w_1, w_2 has in each of them at least one local extremum. Suppose that $\operatorname{ind}(w_1) = \operatorname{ind}(w_2)$. Then $w_1(a) = w_2(a)$, $w_1(b) = w_2(b)$. Since, in addition, each of w_1, w_2 has an extremum in each U_j, it follows that $w_1(t) = w_2(t)$ at $n+1$ points, whence, by the Chebyshev property of $\mathcal{M}$, w_1 and w_2 coincide identically.

Thus it has been proved that $\mathfrak{B}$ is mapped onto K_n, and this mapping is locally invertible at every interior point of $K_n^{\pm}$. It remains to consider the assertion of uniqueness, which, in our terminology, means that on the set of interior points of $K_n^{\pm}$ the mapping ind maps $\mathcal{M}$ one-to-one. Suppose, to the contrary, that some interior point of K_n^{+} has more than one inverse image. Then, by the preceding, every point of $\Delta_n \setminus \Delta_{n-1}$ has the same number of inverse images. But consider $\omega = (0, +, 1, \dots, 1)$ (the coordinate 1 occurring n times). From the assumption that w_1 and w_2 are distinct elements of $\mathcal{M}$ such that $\operatorname{ind}(w_1) = \operatorname{ind}(w_2) = \omega$, we obtain a contradiction. Let $n = 2m$, and let $\tau_0 = a, \tau_1, \dots, \tau_m$ be the points at which w_1 vanishes; then $w_1 - w_2$ has at least two zeros in $[\tau_{i-1}, \tau_i]$, or vanishes at one point but preserves its sign in passing through it; moreover, $w_1(b) - w_2(b) = 0$. Then, taking double zeros into account, $w_1 - w_2$ has $\geq n+1$ zeros, which is impossible for an element of a Chebyshev system (see ([1],[2])). The argument is analogous for odd n. The proof is complete.

We note the following result, which is obtained by arguments similar to those used in proving Lemma 3 and which, apparently, is of independent interest.

Proposition 4. *For any integer m, the set of functions $w \in C[a,b]$ having trees with at least m edges is open.*

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REFERENCES

1. M. G. Krein, UMN, 6, No. 4, 3 (1951).
2. S. Karlin, W. Studden, Tchebycheff Systems, N. Y., 1966.
3. A. C. Korpord, UMN, 5, No. 1, 24 (1950).
4. A. J. Kempner, Tohôku Math. J., 37, 347 (1933).
5. A. J. Kempner, Tohôku Math. J., 42, 318 (1936).
6. Ch. Davis, Am. Math. Monthly, 64, No. 9, 679 (1957).
7. J. Mycielski, S. Paszkowski, Bull. Acad. Polon. Sci., Sér. Math., Astronom., Phys., 8, No. 7, 433 (1960).
8. W. J. Kammerer, Math. Comput., 15, No. 74, 115 (1961).
9. J. W. Andrushkiw, Bull. Am. Math. Soc., 62, 243 (1956).
10. G. R. MacLane, Trans. Am. Math. Soc., 62, 99 (1947).
11. R. S. Johnson, Trans. Am. Math. Soc., 96, 458 (1960).
12. V. S. Videnskii, DAN, 162, No. 2, 251 (1965).
13. Ch. Davis, Analyse fonctionnelle et approximation de fonctions, Duplicated lecture notes, Canadian Math. Soc., 1966.
14. V. S. Videnskii, DAN, 171, No. 1, 17 (1966).

Note: Figure translations are in progress. See original paper for figures.