UNITARY EXTENSIONS OF ISOMETRIC OPERATORS IN THE PONTRYAGIN SPACE \(\Pi_1\) AND EXTENSIONS IN THE CLASS \(\mathfrak{P}_1\) OF FINITE SEQUENCES OF THE CLASS \(\mathfrak{P}_{1;n}\)

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 517.948:513.88

MATHEMATICS

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UNITARY EXTENSIONS OF ISOMETRIC OPERATORS IN THE PONTRYAGIN SPACE Π_1 AND EXTENSIONS IN THE CLASS $\mathfrak{P}_1$ OF FINITE SEQUENCES OF THE CLASS $\mathfrak{P}_{1;n}$

(Presented by Academician P. S. Novikov on 30 V 1966)

1. A finite sequence of complex numbers $\{c_p\}_0^{n-1}$ $(\bar{c}_0 = c_0)$ belongs, by definition, to the class $\mathfrak{P}_{\chi;n}$, if the Toeplitz form

$$\sum_{p,q=0}^{n-1} c_{p-q}\xi_p\bar{\xi}_q \quad (c_{-p} = \bar{c}_p;\ p = 0, 1, \dots, n-1) \tag{1}$$

has exactly χ (≥ 0) positive squares. If

$$\Delta_{n-1} = \det \|c_{p-q}\|_{p,q=0}^{n-1} \neq 0$$

(a nondegenerate sequence), the sequence $\{c_p\}_0^{n-1}$ can be extended in infinitely many ways to a sequence $\{c_p\}_0^{\infty}$ of the class $\mathfrak{P}_\chi$, i.e. such that all forms (1), for arbitrarily large n, preserve exactly χ positive squares (([1]), Ch. V).

In the case where $\chi = 1$, the sequences $\{c_p\}_0^{\infty} \in \mathfrak{P}_1$ may be of three types, according as $\{c_p\}_0^{\infty}$ is bounded (elliptic type), and, in the case of unboundedness, according as the finite or infinite (necessarily existing) limit

$$\lim_{p\to\infty} (|c_p|/p^2)$$

exists (parabolic and hyperbolic types, respectively) ([1,2]). In this connection there naturally arises the question of the existence, for a finite sequence $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ $(\Delta_{n-1} \neq 0)$, of infinite extensions of one or another type in the class $\mathfrak{P}_1$. Some of the very first results in this direction for real extensions of real sequences $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ were contained in ([1–3]), and for complex sequences $\{c_p\}_0^{1} \in \mathfrak{P}_{1;2}$—in ([2]).

In the present note the problem under consideration is connected with the more general problem of extending isometric operators in the Pontryagin space Π_χ

(and, in particular, Π_1) to unitary operators in the same or in a wider space of the same type*.

2. We begin by formulating (with a certain refinement) a theorem whose second part, in essence, was already contained in the reasoning of § 9 (Ch. III) of (^{1}).

Theorem 1. *In order that an isometric operator V in the space Π_χ admit unitary extensions in this same space Π_χ or in some wider space** $\widetilde{\Pi}_\chi \supset \Pi_\chi$, it is necessary and sufficient that V be bounded and continuously invertible. Unitar–*

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* For definitions of all notions occurring here and below, connected with the geometry of the spaces Π_χ and operators acting in these spaces, see, for example, (^{1}).

** The embedding $\Pi_\chi \to \widetilde{\Pi}_\chi$ is isometric. The necessity of the requirement of boundedness and continuous invertibility of the operator V for the existence of its unitary extensions was first clarified in (^{4}).

extensions without exit from $\Pi_\varkappa$ are possible if and only if the defect numbers of the operator V coincide.

Let now $\varkappa = 1$. According to a well-known theorem (^{1}), a unitary operator U in the space Π_1 always has a nonnegative eigenvector: $Uf_0 = \lambda_0 f_0$, $f_0 \neq 0$, $(f_0, f_0) \geq 0$, where (f, g) is the indefinite inner product specified in Π_1. If the operator U has a positive eigenvector f_0, then necessarily $|\lambda_0| = 1$ for the corresponding eigenvalue λ_0. Such an operator U will be called **elliptic**. If U is not elliptic, but there exists an eigenvector $f_0 : Uf_0 = \lambda_0 f_0$ with $(f_0, f_0) = 0$ and $|\lambda_0| = 1$, then U is called **parabolic**. Finally, if $Uf_0 = \lambda_0 f_0$ and $|\lambda_0| \neq 1$, then necessarily $(f_0, f_0) = 0$, and there exists a neutral eigenvector f_0': $(f_0', f_0') = 0$, $(f_0, f_0') = 1$, $Uf_0' = \bar{\lambda}_0^{-1} f_0'$, while the operator U is called **hyperbolic**. The three cases listed exclude one another and exhaust all possibilities (^{1}).

The uniquely determined eigenvalues λ_0 appearing in the definitions given above will be called **critical** (cf. (^{5})), and the pair $\{\lambda_0, \bar{\lambda}_0^{-1}\}$ (in the hyperbolic case) will be called the **critical pair** for the operator U.

3. For an arbitrary complex λ and any linear operator T acting in the space Π_1 with domain of definition $\mathfrak{D}_T$, define the subspace

$$\mathfrak{M}_\lambda(T) = \{Tf - \lambda f\}_{f \in \mathfrak{D}_T}.$$

In Theorems 2 and 3 below, where unitary extensions of an isometric operator $V(\mathfrak{D}_V \subset \Pi_1)$ are discussed, extensions without exit from Π_1 are meant if the defect numbers of the operator V are equal, and extensions with exit into $\widetilde{\Pi}_1 (\supset \Pi_1)$ otherwise.

Theorem 2. In order that a bounded and continuously invertible isometric operator $V(\mathfrak{D}_V \subset \Pi_1)$ admit an elliptic unitary extension with critical number ε $(|\varepsilon| = 1)$, it is necessary and sufficient that the subspace $\mathfrak{M}_\varepsilon(V)$ be negative or equal to $\{0\}$.

Theorem 3. In order that a bounded and continuously invertible operator $V(\mathfrak{D}_V \subset \Pi_1)$ admit a hyperbolic unitary extension U with critical pair $\{\lambda_0, \bar{\lambda}_0^{-1}\}$ $(|\lambda_0| \neq 1)$, it is necessary and sufficient that there exist a pair of neutral vectors $f_0, f_0' \in \Pi_1$: $(f_0, f_0) = (f_1, f_1) = 0$, $(f_0, f_0') = 1$, such that

$$f_0 \perp \mathfrak{M}_{\bar{\lambda}0^{-1}}(V), \qquad f_0' \perp \mathfrak{M}_{\lambda_0}(V).$$

If, however, the operator V has no eigenvectors, then a sufficient condition is the degeneracy of the metric on the subspace $\mathfrak{M}_{\lambda_0}(V)$.

The proofs of Theorems 1-3 (in their "sufficiency" part) contain a description of the procedure for obtaining all extensions of the corresponding type. We do not give here (because of its bulkiness) the criterion we also obtained for the existence of a parabolic extension with a prescribed critical number ε $(|\varepsilon| = 1)$.

4. The criteria formulated in Theorems 1-3 (essentially purely geometric ones) make it possible to obtain answers to the questions posed in item 1. Indeed, as was shown in Ch. V $(^4)$, with every infinite sequence $\{c_p\}_0^\infty \in \mathfrak{P}_\varkappa$ there are associated a certain space $\Pi_\varkappa$ and a unitary operator U $(U\Pi_\varkappa = \Pi_\varkappa)$ such that, for some vector $e_0 \in \Pi_\varkappa$, we have $(U^p e_0, e_0) = c_p$ $(p = 0, 1, 2, \ldots)$. In a similar way, with every nondegenerate finite sequence $\{c_p\}_0^{n-1} \in \mathfrak{P}_{\varkappa;n}$ there is associated a finite-dimensional space $\Pi_\varkappa^{(n)}$ $(\dim \Pi_\varkappa^{(n)} = n)$ and in it an isometric "shift" operator V with defect index $(1, 1)$, possessing the property that, for some $e_0 \in \Pi_\varkappa^{(n)}$, we have $c_p = (V^p e_0, e_0)$ $(p = 0, 1, \ldots, n-2)$. Thus, the problems of existence and classification of extensions of nondegenerate sequences $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n-1}$ reduce to the corresponding problems of extension of isometric-

···operators to unitary ones (without leaving or with leaving $\Pi_1^{(n)}$). In the case of extensions without leaving $\Pi_1^{(n)}$ we shall obtain sequences $\{c_p\}_0^\infty \in \mathfrak{P}_1$ of finite rank n $(^1)$, i.e., for them

$$\det \|c_{p-q}\|_{p,q=0}^m = 0 \quad (m \geqslant n).$$

The critical numbers obtained with the aid of the above-mentioned procedure of unitary extensions then occur in the integral representations of the extended sequences $\{c_p\}_0^\infty$ (see $(^1)$, Chap. V); in view of this, these numbers may also be called the **critical numbers of the sequences** $\{c_p\}_0^\infty$.

Theorem 4. *A nondegenerate sequence* $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ *admits extensions of elliptic type* $\{c_p\}_0^\infty \in \mathfrak{P}_1$ *with critical number* ε $(|\varepsilon| = 1)$ *if and only if the* *sequence* $d_p = L_\varepsilon[c_p]$ $(p = 0, 1, \dots, n-2)$ *is (strictly) positive* (^{6})*; moreover exactly one of these extensions will have rank* n*.*

Let us explain that here by L_λ, where $\lambda(\neq 0)$ is an arbitrary complex number, we denote a finite-difference (so-called defining) operator (cf. (^{1})), acting by the formula

$$L_\lambda[c_p] = \bar{\lambda} c_{p+1} - (1 + |\lambda|^2) c_p + \lambda c_{p-1} \quad (p = 0, 1, \dots, n-2).$$

As examples show, there exist finite nondegenerate sequences of the class $\mathfrak{P}_{1;n}$ which admit no extensions of elliptic type at all; this is easily checked by means of Theorem 4. For example, for $n = 3$ * such a sequence is $\{0, 1, 4\} \in \mathfrak{P}_{1;3}$.

Theorem 5. *In order that a nondegenerate sequence* $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ *admit an extension* $\{c_p\}_0^\infty \in \mathfrak{P}_1$ *of parabolic type with critical number* ε $(|\varepsilon| = 1)$*, it is necessary that the sequence* $\{L_\varepsilon[c_p]\}_0^{n-2}$ *be nonnegative* (^{6})*. The degeneracy of this sequence is necessary and sufficient for the existence of the corresponding parabolic extension of rank* n*, which (for fixed* ε*) is unique.*

Application of the last criterion leads to the following result:

Theorem 6. *A nondegenerate sequence* $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ *admits no more than* $2(n-1)$ *distinct parabolic extensions* $\{c_p\}_0^\infty \in \mathfrak{P}_1$ *of rank* n*; the critical numbers* ε *of these extensions are found from the equation*

$$\det \|L_\varepsilon[c_{p-q}]\|_{p,q=0}^{n-2} = 0.$$

For $n = 2$ there always exist exactly two distinct parabolic extensions of rank 2. Examples show that for $n > 2$ a sequence $\{c_p\}_0^{n-1} \in \mathfrak{P}_1$ may have no parabolic (and at the same time no elliptic—cf. below with Theorem 8) extensions at all. Such, for example, is the sequence $\{0, 1, 5\} \in \mathfrak{P}_{1;3}$. There exist examples of sequences $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ admitting parabolic extensions of rank n in number k, where k is any integer, $0 \leqslant k \leqslant 2(n-1)$.

Theorem 7. *Every nondegenerate sequence* $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ *admits hyperbolic extensions* $\{c_p\}_0^\infty \in \mathfrak{P}_1$*, including a continual family (depending on one real parameter) of extensions of rank* n*, whose critical numbers (i.e., critical pairs* $\{\lambda, \bar{\lambda}^{-1}\}$*,* $0 < |\lambda| \neq 1$*) are determined from the equation*

$$\det \|L_\lambda[c_{p-q}]\|_{p,q=0}^{n-2} = 0.$$

Our proof of Theorem 7 is based, in particular, on a recently discovered rule for computing the signature of an arbitrary Hermitian Toeplitz form (^{7}). Closely adjacent to Theorem 7 is the simpler

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* We note that for $n = 2$ a nondegenerate sequence $\{c_p\}_0^1$ of class $\mathfrak{P}_{1;2}$ always admits extensions $\{c_p\}_0^\infty \in \mathfrak{P}_1$ of rank 2 of all three types (([2]), Theorem 4). This fact is now obtained as a very special consequence of the theorems of the present work.

Theorem 7. In order that a nondegenerate sequence $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ admit extensions of hyperbolic type with critical pair $\{\lambda, \bar{\lambda}^{-1}\}$ $(0 < |\lambda| \neq 1)$, it is necessary that the sequence $\{L_\lambda[c_p]\}_0^{n-2}$ be nonnegative. The degeneracy of this sequence is necessary and sufficient for the existence of the corresponding hyperbolic extension of rank n, which (for fixed λ) is determined uniquely.

Theorem 8. If a nondegenerate sequence $\{c_p\}_0^{n-1} \in \mathfrak{P}_{1;n}$ has an "elliptic extension," then there are infinitely many extensions of this type, among which the extensions of rank n form a nonempty family depending on one parameter ranging over a certain open set on the real axis. In this case the sequence admits extensions (including those of rank n) of all three types.

In conclusion we note that Theorems 4–8 are essentially equivalent to the corresponding facts in the theory of extensions of isometric "shift" operators in finite-dimensional spaces Π_1. We shall not dwell here on possible generalizations of the theory developed above to the case $\chi > 1$.

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Received
26 V 1966

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[1] I. S. Iokhvidov, M. G. Krein, *Tr. Mosk. Matem. Obshch.*, **5**, 169 (Chapters I-III) (1956); **8**, 413 (Chapters IV–VI) (1959).
[2] I. S. Iokhvidov, *Dokl. AN ArmSSR*, **42**, No. 3 (1966).
[3] I. S. Iokhvidov, M. G. Krein, *Tr. Mosk. Matem. Obshch.*, **15** (1966).
[4] I. S. Iokhvidov, *UMN*, **16**, 4 (100), 175 (1961).
[5] M. G. Krein, G. K. Langer, *DAN*, **152**, 1, 39 (1963).
[6] N. I. Akhiezer, M. G. Krein, *On Some Questions in the Theory of Moments*, Kharkov, 1938.
[7] I. S. Iokhvidov, *DAN*, **169**, No. 6, 1258 (1966).

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