CAUCHY–RIEMANN AND LEBESGUE INTEGRALS. INTERMEDIATE INTEGRALS

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Abstract

Full Text

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MATHEMATICS

P. P. KOROVKIN

CAUCHY–RIEMANN AND LEBESGUE INTEGRALS. INTERMEDIATE INTEGRALS

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Let $f(x)$ be a function bounded on the interval $[a, b]$, and let $0 < \varepsilon \leq 1/2$. Denote by $A_{h,\delta,x}$ the set of points of the interval $[a, b]$ in the neighborhood $|t - x| < \delta$ at which $f(t) \geq h$,

$$A_{h,\delta,x} = \{t,\ f(t) \geq h\} \cap \{|t - x| < \delta\} \cap [a, b].$$

The set $B_{h,\delta,x}$ is defined analogously:

$$B_{h,\delta,x} = \{t,\ f(t) \leq h\} \cap \{|t - x| < \delta\} \cap [a, b].$$

Put

$$M(x, \varepsilon, \delta) = \inf_{m^* A_{h,\delta,x} < \varepsilon \cdot 2\delta} h, \qquad m(x, \varepsilon, \delta) = \sup_{m^* B_{h,\delta,x} < \varepsilon \cdot 2\delta} h,$$

where $m^* E$ is the outer Lebesgue measure of the set E.

Theorem 1. *Almost everywhere on the interval $[a, b]$ there exist the limits*

$$M(x) = \lim_{\delta \to 0} M(x, \varepsilon, \delta), \qquad m(x) = \lim_{\delta \to 0} m(x, \varepsilon, \delta),$$

which do not depend on ε, $0 < \varepsilon \leq 1/2$. The functions $M(x)$ and $m(x)$ are measurable and almost everywhere $M(x) \geq f(x) \geq m(x)$. If the function $f(x)$ is measurable, then almost everywhere $M(x) = m(x)$.

Let now T be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \dots < x_n = b$, $\lambda(T) = \max(x_{k+1} - x_k)$; $E_{k,h}$ is the set of points of the interval $[x_k, x_{k+1}]$, $k = 0, 1, \dots, n - 1$, at which $f(x) \geq h$; $H_{k,h}$ is the set of points of the same interval at which $f(x) \leq h$, and

$$M_k(\varepsilon) = \inf_{m^* E_{k,h} \leq \varepsilon \Delta x_k} h, \qquad m_k(\varepsilon) = \sup_{m^* H_{k,h} < \varepsilon \Delta x_k} h,$$

$$\Delta x_k = x_{k+1} - x_k, \qquad 0 < \varepsilon \leq 1/2.$$

Consider the ε-upper and ε-lower integral sums of the function $f(x)$:

$$\sum_{k=0}^{n-1} M_k(\varepsilon) \Delta x_k, \qquad \sum_{k=0}^{n-1} m_k(\varepsilon) \Delta x_k.$$

Theorem 2. *The equalities hold*

$$\lim_{\lambda \to 0} \sum_{k=0}^{n-1} M_k(\varepsilon) \Delta x_k = \int_a^b M(x)\, dx,$$

$$\lim_{\lambda \to 0} \sum_{k=0}^{n-1} m_k(\varepsilon) \Delta x_k = \int_a^b m(x)\, dx,$$

where the integrals are understood in the Lebesgue sense.

We shall call the first integral the upper ε-integral of the function $f(x)$,

and the second the lower. We shall call the function $f(x)$ ε-integrable if the two limits coincide,

$$\int_a^b m(x)\, dx = \int_a^b M(x)\, dx.$$

Corollary. For ε-integrability it is necessary and sufficient that the function $f(x)$ be measurable in the sense of Lebesgue, and the ε-integral coincides with the Lebesgue integral.

Thus, starting from integral sums similar to Riemann sums, the Lebesgue integral is obtained.

Let us now suppose that the quantity ε participating in the construction of the integral sums is not constant, but is a function of the argument λ, where λ, as before, is the length of the largest of the subintervals of the partition T, $\lambda = \max_{k=0,1,\ldots,n-1} \Delta x_k$. Consider the sums

$$\sum_{k=0}^{n-1} M_k[\varepsilon(\lambda)]\, \Delta x_k, \qquad \sum_{k=0}^{n-1} m_k[\varepsilon(\lambda)]\, \Delta x_k, \qquad 0 < \varepsilon(\lambda) \leq \tfrac{1}{2}.$$

The function $f(x)$ for which the limits of both sums as $\lambda \to 0$ are the same will be called integrable, and the class of integrable functions will be denoted by $R[\varepsilon(\lambda)]$.

Theorem 3. If a bounded function $f(x)$ is measurable, then there exists a function $\varepsilon(\lambda)$, $\lim_{\lambda\to 0} \varepsilon(\lambda) = 0$, such that

$$f(x) \in R[\varepsilon(\lambda)].$$

Corollary. If $\mathcal{L}$ is the set of bounded and measurable functions, then

$$\mathcal{L} = \bigcup_{\varepsilon(\lambda)=o(1)} R[\varepsilon(\lambda)].$$

Theorem 4. Whatever the function $\varepsilon(\lambda)$ may be, $\varepsilon(\lambda) = o(1)$, there exists a measurable and bounded function $f(x)$ such that

$$f(x) \in R[\varepsilon(\lambda)].$$

Corollary. The class $R[\varepsilon(\lambda)]$ is strictly intermediate between the class R of Riemann-integrable functions and the class $\mathcal{L}$ of Lebesgue-integrable functions,

$$R \subset R[\varepsilon(\lambda)] \subset \mathcal{L} = R(1/2).$$

Let now $\varepsilon_1(\lambda)$ and $\varepsilon_2(\lambda)$ be two positive functions monotonically tending to zero when $\lambda \to 0$. If

$$\lim_{\lambda\to 0} \frac{\varepsilon_1(\lambda)}{\varepsilon_2(\lambda)} = 0, \tag{1}$$

then it is clear that $R[\varepsilon_1(\lambda)] \subseteq R[\varepsilon_2(\lambda)]$, since the corresponding functions $\varepsilon_1(\lambda)$-integral sums occupy an intermediate position among the sums for the function $\varepsilon_2(\lambda)$.

Theorem 5. If equality (1) holds, then there exists a function $f(x)$

$$f(x) \in R[\varepsilon_1(\lambda)], \qquad f(x) \notin R[\varepsilon_2(\lambda)],$$

i.e.

$$R[\varepsilon_1(\lambda)] \subset R[\varepsilon_2(\lambda)].$$

Thus, there exist infinitely many integrals which, with respect to the class of integrable functions, occupy an intermediate position between the Riemann and Lebesgue integrals. At the same time, there also exist infinitely many measures of a linear set, if by measure one understands the integral of the characteristic function of the set.

It is known that every function bounded on an interval $[a, b]$ and discontinuous inside the interval has an antiderivative. As for functions

discontinuous ones, then for them there is no such sufficient condition. The necessary conditions for the derivative are present. The derivative is a measurable function, passes through all intermediate values, etc.

Theorem 6. If the limits specified in the condition of Theorem 1 exist everywhere inside the interval $[a, b]$, and everywhere inside the interval $m(x) = f(x) = M(x)$, then there exists an antiderivative function $f(x)$.

It is known that the class of bounded derivatives constitutes a proper part of the class of measurable and bounded functions. The intermediate integral over the interval $[a, x]$ restores the antiderivative if only it exists from the derivative,

$$\int_a^x f'(x)\,dx = f(x) - f(a).$$

Theorem 7. Whatever the function $\varepsilon(\lambda)$, $\varepsilon(\lambda) = o(1)$, there exists a bounded $f'(x)$, $f'(x) \in R[\varepsilon(\lambda)]$.

Thus, only the Lebesgue integral restores the antiderivatives of all bounded derivatives, since $f'(x) \in \mathcal{L} = R(1/2)$.

Moscow Automobile and Road Institute

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Note: Figure translations are in progress. See original paper for figures.