The existence of optimal controls in systems with lag

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Abstract

An existence theorem for an optimal control that minimizes the functional $J(u) = \Phi(x(t_1))$ on the trajectories of the system

$$\dot{x}(t) = f(x(t), x(t - h(x(t), u(t), t)), u(t), t), \quad x(\tau) = \varphi(\tau), \quad \tau \in S_0.$$

is proved. Bibliography: 5 items.

Full Text

Preamble

This paper considers a dynamical system described by the following functional differential equation with a state-dependent delay:

$$\frac{dx(t)}{dt} = f[x(t), x(t - h(x,t)), u(t), t], \quad t \in T = [t_0, t_1]$$

subject to the initial conditions $x(\tau) = \phi(\tau)$ for $\tau \in S_0$, where S_0 is the initial set. Here, $x = (x_1, \dots, x_n)$ is the state vector in E^n, $u = (u_1, \dots, u_r)$ is the control vector belonging to a compact set $U \subset E^r$, and $h(x,t)$ is the delay function.

We assume the following conditions hold: 1. The function $f(x, y, u, t)$ is defined on the domain $P = X \times X \times U \times T$ and satisfies a Lipschitz condition:

$$\|f(x', y', u, t) - f(x, y, u, t)\| \le L_1(\|x' - x\| + \|y' - y\|)$$

2. The delay function $h(x,t)$ is defined on $Q = X \times T$ and satisfies:

$$|h(x', t) - h(x, t)| \le L_2 \|x' - x\|$$

3. The initial function $\phi(t)$ is defined on S_0 and is Lipschitz continuous:

$$\|\phi(t') - \phi(t)\| \leq L_3|t' - t|$$

4. The function f is bounded such that $\|f(x, y, u, t)\| \leq L_4$ for all points in P.

1. Existence and Uniqueness of Solutions

To prove the existence of a solution $x(t)$ on the interval $[t_0, t^*]$, we employ the method of successive approximations. Let $x_0(t) = \phi(t_0)$ for $t \in [t_0, t^*]$ and define the sequence:

$$x_{m+1}(t) = \phi(t_0) + \int_{t_0}^{t} f(x_m(\tau), x_m(\tau - h(x_m(\tau), \tau)), u(\tau), \tau)d\tau$$

where $x_m(\tau) = \phi(\tau)$ if $\tau \in S_0$. For a sufficiently small interval $[t_0, t^*]$, where $t^* = t_0 + \Delta$, the sequence $\{x_m(t)\}$ converges uniformly to a unique solution $x(t)$. The error estimate for the m-th approximation is given by:

$$\|x(t) - x_m(t)\| \leq 2bL^m(2 + L_2)^{m-1}\frac{(t - t_0)^m}{m!}$$

where b is a constant related to the initial bounds of the function.

2. Stability and Continuous Dependence

We further analyze the dependence of the solution on the control $u(t)$ and the initial function $\phi(t)$. If $x(t)$ and $\bar{x}(t)$ are solutions corresponding to controls $u(t)$ and $\bar{u}(t)$, then for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\int_{t_0}^{t_1} \|u(\tau) - \bar{u}(\tau)\|d\tau < \delta$, then $\|x(t) - \bar{x}(t)\| < \epsilon$ for all $t \in T$.

Specifically, the difference between solutions can be bounded as:

$$\|x(t) - \bar{x}(t)\| \leq (M_1 + ML_2)\delta e^{L_1(2+ML_2)(t-t_0)}$$

where $M = \max\{L_3, L_4\}$. This demonstrates that the system is stable with respect to small perturbations in the control input and initial data.

3. Optimal Control and Differential Inclusions

Consider the problem of minimizing a functional $J(u) = \Phi(x(t_1))$. To analyze the set of reachable states, we define the differential inclusion:

$$\frac{dx(t)}{dt} \in R(x(\cdot), t)$$

where $R(x(\cdot), t) = \{z : z = f[x(t), x(t - h(x(t), t)), u, t], u \in U\}$.

The set of trajectories $X_t(\cdot)$ is compact in the space of continuous functions. For any trajectory $z(t)$ of the relaxed system (the convex hull of the velocity set), there exists a sequence of trajectories of the original system (1) that converges uniformly to $z(t)$. This allows us to apply the Filippov-type existence theorems for optimal control.

4. Necessary Conditions for Optimality

Using the properties of the reachable set and the continuity of the mapping $R(x(\cdot), t)$, we can derive necessary conditions for the optimal control $u(t)$. If $u^*(t)$ is an optimal control, then there exists a non-zero adjoint vector function $\psi(t)$ satisfying the corresponding adjoint system, such that the Hamiltonian reaches its maximum:

$$H(x, \psi, u, t) = \max_{u \in U} (\psi(t), f[x, y, u, t])$$

The presence of the state-dependent delay $h(x, t)$ introduces additional terms into the adjoint equation involving the derivative of the delay with respect to the state x.

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Figures

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DIFFERENTIAL EQUATIONS

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RECURRENT SOLUTIONS OF DIFFERENTIAL EQUATIONS AND THE GENERAL THEORY OF DYNAMICAL SYSTEMS

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In the present note, such solutions of differential equations, called *fully recurrent* below (see definition 2), which generate recurrent motions in the sense of Birkhoff [1—4] in the dynamical system of shifts of continuous functions are studied. The mentioned nassome dynamical system is a generalization of the universal dynamical system of Bebutov [5, 6] to the case where the phase space consists of continuous mappings of the real - line into an arbitrary metric space. By their properties on poousouscoce me-trumeckoe poostranctoo.

By their properties, fully recurrent functions are more general than almost periodic ones (in the sense of Bohr [7]), and are closely related to recurrent functions (see definition 1). So, for example, for a differential equation deter-dynamical system, the concepts of a recurrent and fully recurrent solution are equivalent.

The problem of studying solutions of differential equations generating recurrent motions in the dynamical system of Bebutov was posed by V. V. Nemytskii ([3], p. 101).

One of the questions considered in this note can be formulated as follows: under what conditions from the existence of a solution stable according to Lagrange in the positive direction of the differential equation (13) (see definition 5) does the existence of fully recurrent solutions of this equation follow?

An analogous problem for almost periodic solutions of a narrower class of differential equations was considered in [8] and [9].

If the right-hand side of equation (13) does not depend on the variable t and the conditions of existence, uniqueness, and continuability of solu-solutions on $(-\infty, +\infty)$ are satisfied, then the answer to the posed ques-quation follows from the known theorem of Birkhoff, according to which in the ω-limit set of every stable according to Lagrange in the positive direction motion there exists a minimal set of recurrent motions.

In the general case, the answer to the posed question follows from Theorem 2 proved below (corollaries 2 and 3). From it in an obvious way also follows the sign of existence of recurrent solutions of a non-autonom-ous differential equation in n-dimensional space proved in [9]. On the right-hand side of this equation conditions are imposed, which actually ensure the existence not only of recurrent, but also of fully recurrent solutions.

Another question, considered in this note, can be formulated as follows: under what conditions is a given solution of equation (13) fully recurrent? The answer to this question is given by the here proved Theorem 3 and corollary 4. On the basis of these propositions a recurrent om.u sammım a simple sign of existence of a unique fully recurrent is established

Figure 1: Figure 1

of the linear nonhomogeneous system of differential equations (Theorem 4).

The works [10—12] and partially the work [9] are devoted to the study of recurrent solutions of differential equations. The main results of the present article are outlined without proofs in [12].

Below, just as in [12], for the study of the solutions of equation (13), the transition to a dynamical system of shifts is applied. This method consists in the following: the solutions of the given equation are considered as points of the phase space of a dynamical system in the space of continuous functions, for which the motions are defined as shifts of functions. By applying the methods of the general theory of dynamical systems, the movements that are generated by the solutions of the given equation are studied. Tlus. These movements fully characterize the properties of the solutions under consideration.

The described method of investigating the solutions of differential equations gives the opportunity to use the results of the well-developed-at-present-time general theory of dynamical systems, which leads to sufficiently general results with comparatively simple proofs.

1°. Let us introduce the following notation: T — the real line; T_+ — the interval $[0, +\infty)$; T_- — the interval $(-\infty, 0]$; N — the set of all natural numbers; ρ — the distance in metric spaces*).

Let X and Y be metric spaces.

By the symbol $(X; Y)$ we shall denote the set of all continuous mappings from X into Y, endowed with the *compact-open topology* [13], which is also called the k-topology [14] or the topology of compact convergence [15].

Let φ be a continuous mapping T into Y and $\tau \in T$. By the symbol φ^τ we shall denote the mapping $t \to \varphi(t + \tau)$ of the real line T into Y, by σ — the mapping $(\varphi, t) \to \varphi^t$ of the product $(T; Y) \times T$ into $(T; Y)$, and by the symbol $\sigma(T; Y)$ — the dynamical system in the space $(T; Y)$, defined by the mapping σ (see [4—6]).

Let f be a continuous mapping $T \times X$ into Y. By the symbol f^* we shall denote the mapping $t \to f_t$ of the real line T into $(X; Y)$, where f_t is the partial mapping defined by the function f at the value t of the first argument (see [16]).

2°. Let us introduce some definitions and a series of auxiliary propositions connected with them.

Let X be a compact metric space, Y — a complete metric space.

The spaces $(T; Y)$, $(X; Y)$ and $(T \times X; Y)$ are metrizable (see [15], p. 34). In this connection, in what follows we shall assume that the metric in the spaces $(T; Y)$, $(X; Y)$ and $(T \times X; Y)$ is defined respectively by the formulas

$$\rho(\varphi, \psi) = \sup_{l>0} \min\left\{ \sup_{|t| \leqslant l} \rho(\varphi(t), \psi(t)), \frac{1}{l} \right\}, \qquad \rho(\varphi, \psi) = \sup_{x \in X} \rho(\varphi(x), \psi(x)), \tag{1}$$

*) Under the symbol $\rho(x, y)$ one should understand the distance between x and y in that metric space, the points of which are x and y. The fact that the distance function is denoted by one and the same letter for all metric spaces in what follovnot lead to any ambiguity, since every time when writing $\rho(x, y)$, there will be indicated a completely determined metric space, the points of which are x and y.

Figure 2: Figure 2

$$\rho(\phi, \psi) = \sup_{l>0} \min\left\{ \sup_{|t|<l,\, x\in X} \rho(\phi(t, x), \psi(t, x)), \frac{1}{l} \right\}.$$

Formula (1) was introduced by M. V. Bebutov [5, 6] for the metrication of the set of all continuous mappings of the line into the line. Just as in [5, 6], it can be shown that whatever the points ϕ and ψ of the space $(T; Y)$ may be and whatever the positive number ε may be, the inequality $\rho(\phi, \psi) < \varepsilon$ $(\rho(\phi, \psi) < \varepsilon)$ is satisfied if and only if

$$\sup_{|t|<\frac{1}{\varepsilon}} \rho(\phi(t), \psi(t)) < \varepsilon \qquad (\sup_{|t|<\frac{1}{\varepsilon}} \rho(\phi(t), \psi(t)) < \varepsilon).$$

Lemma 1. The mapping $f \to f_*$ of the space $(T \times X; Y)$ into $(T; (X; Y))$ is an isometry of the space $(T \times X; Y)$ onto $(T; (X; Y))$.

Proof. The mapping indicated in the lemma is a homeomorphism of the space $(T \times X; Y)$ onto $(T; (X; Y))$ (see [15], p. 46, corollary 2).

Let f and g be points of the space $(T \times X; Y)$. Whatever $l > 0$ may be, the equality holds

$$\sup_{|t|<l,\, x\in X} \rho(f(t, x), g(t, x)) = \sup_{|t|<l} \rho(f_*(t), g_*(t)) \tag{2}$$

Indeed,

$$\sup_{|t|<l} \rho(f_*(t), g_*(t)) = \sup_{|t|<l} \sup_{x\in X} \rho(f(t), g_*(t)) =$$
$$= \sup_{|t|<l} \sup_{x\in X} \rho\big(f(t, x), g(t, x)\big) = \sup_{|t|<l,\, x\in X} \rho\big(f(t, x), g(t, x)\big).$$

From (2) it follows that $\rho(f, g) = \rho(f_*, g_*)$. Lemma proved.

Let φ be a continuous mapping of T into Y.

Definition 1. A function φ is called *recurrent* if for any $\varepsilon > 0$ there exists $l > 0$ such that for every $\tau_0 \in T$ on any interval of length l there can be found a number τ for which

$$\rho(\varphi(\tau), \varphi(\tau_0)) < \varepsilon.$$

In the case when φ is a motion under some dynamical system defined in the space Y, the definition formulated above coincides with the definition of recurrent motion in the sense of Birkhoff [1], and in the case when φ is a real function, with the definition given in [17].

Definition 2. A function φ is called *almost recurrent* if for any $\varepsilon > 0$ there exists $l > 0$ such that for every $\tau_0 \in T$ on any interval of length l there can be found a number τ for which

$$\sup_{|t|<\frac{1}{\varepsilon}} \rho(\varphi(t+\tau), \varphi(t+\tau_0)) < \varepsilon.$$

It is clear that every almost recurrent function is recurrent. The converse statement does not hold. Thus, for example, the function

$$\omega(t) = \begin{cases} 0, & t \in [0, \pi], \\ \sin t, & t \in (\pi, 2\pi), \end{cases}$$

considered in [17], is recurrent, but not almost recurrent.

However, if φ is a motion under some dynamical system, defined

Figure 3: Figure 3

in space Y, then definitions 1 and 2 are equivalent. This follows from the fact that on the trajectory of recurrent motion, contained in a complete space, the property of integral continuity is fulfilled uniformly.

Lemma 2. The following conditions are equivalent:

1) The partial mapping σ_φ, given by the mapping $\sigma : (T; Y) \times T \to (T; Y)$, for the value φ of the first argument is recurrent.

2) The function φ is completely recurrent.

3) For any sequence of numbers $\{\tau_n\}$, one can choose a subsequence $\{\tau_{nk}\}$, a completely recurrent mapping $\psi : T \to Y$ and a sequence $\{\tau'_n\}$ such that the sequences $\{\varphi^{\tau_{nk}}\}$ and $\{\psi^{\tau'_n}\}$ of points of the space $(T; Y)$ converge and the equalities hold

$$\lim_{k\to\infty} \varphi^{\tau_{nk}} = \psi, \quad \lim_{n\to\infty} \psi^{\tau'_n} = \varphi.$$

Proof. The equivalence of conditions 1 and 2 of the lemma follows directly from definitions 1 and 2.

Consider the mapping $\sigma_\varphi : T \to (T; Y)$, which is a motion under the dynamic system $\sigma(T; Y)$. Condition 3 of the lemma is equivalent to the following: the closure of the trajectory of the motion σ_φ is a compact minimal set. The latter condition, in view of the completeness of the space $(T; Y)$ (see [15], p. 20) is equivalent to condition 1 of the lemma (see [2], chap. V, theorems 27, 28).

Using the proved lemma, it is easy to verify that if the fynction φ is completely recurrent, then it is uniformly continuous and the set $\varphi(T)$ is compact in Y. Indeed, if the fynction φ is completely recurrent, then, as follows from condition 3 of the lemma, the motion σ_φ under dynamic system $\sigma(T; Y)$ is stoble according to Lagrange. Since the space $(T; Y)$ is complete, the latter means that the fynction φ is niformly continuous and the set $\varphi(T)$ is compact in Y (see [4], lemma 1.65).

In the following, an interval is called a non-empty set $J \subseteq T$ such that, whatever points a and b are from J, connected by the inequality $a \approx b$, the etgrsent $[a, b]$ is contained in J.

Definition 3. Let φ be a continuous mapponing of the interval J into Y. The fynction φ is called a ω-limit (α-limit) imagearenage of the fynction ψ, if there exists such a numerical sequente condexansmepersance $\{\tau_n\}$, converging to $+\infty$ ($-\infty$), that for any $\varepsilon > 0$ there is such $n_0 \in \mathbb{N}$, that $[-1/\varepsilon+\tau_n, 1/\varepsilon+\tau_n] \subseteq J$ and

$$\sup_{|t|<1\frac{1}{\varepsilon}} \rho\,(\psi(t+\tau_n), \varphi(t)) \leqslant \varepsilon$$

for all natural $n \geqslant n_0$.

From the last definition and the definition of ω-limit (α-limit) point of motion (see [2], chap. V, § 3) it follows that φ is a ω-limit (α-limit) image of the continuous cotoppasseening $\psi : T \to Y$ if and only if φ is a ω-limit (α-limit) point of the nunction σ_φ under the dynamic system $\sigma(T; Y)$.

3°. Let B be a real Banach space. We will prove some criteria for the existence of completely recurrent solutions of differential equations given in space B.

First, consider the differential equation

$$x' = f(t)x, \tag{3}$$

Figure 4: Figure 4

where f is a completely recurrent mapping T in $(D; B)$, and D is some compact of the space B.

Remark 1. Equation (3) is equivalent to the differential equation

$$x' = F(t, x),$$

where F is a mapping of the product $T \times D$ into B, defined by the relation

$$F(t, x) = f(t) \times (t \in T, x \in D).$$

Moreover, according to lemma 1, if the function f is continuous, then F is also continuous.

Theorem 1. *Assume that for the differential equation* (3) *there exists a solution* φ_0, *defined on some unbounded on the right (left) interval* $J = [t_0, +\infty)$ $(J = (-\infty, t_0])$. *Then, whatever the function* g, *belonging to the closure of the set* $\{f_t \mid \tau \in T\}$ *of the space* $(T; (D; B))$, *there exists a completely recurrent solution of the differential equation*

$$x' = g(t)\, x, \tag{4}$$

which is an ω*-limit* (α*-limit*) *image of the function* φ_0.

For the proof of the theorem formulated above, the following will be needed

Lemma 1) for any $n \in \mathrm{N}$ there exists such a continuous mapping $\varphi_n \colon T \to D$ and such an interval $J_n = (a_n, b_n)$, that the restriction φ_n on J_n is a solution of the differential equation

$$x' = h_n(t)\, x;$$

2) $\lim\limits_{n \to \infty} a_n = -\infty$ and $\lim\limits_{n \to \infty} b_n = +\infty$.

Then: the set $\{\varphi_n \mid n \in \mathrm{N}\}$ is compact in the space $(T; D)$ and the limit of any convergent subsequence of the sequence $\{\varphi_n\}$ of points of the space $(T; D)$ is a solution of the differential equation

Lemma 3. *Let* $\{h_n\}$ *be a convergent sequence of points of the space* $(T; (D; B))$ *and*

$$g = \lim_{n \to \infty} h_n. \tag{5}$$

Assume: 1) for any $n \in \mathbb{N}$ there exists such a continuous mapping $\varphi_n \colon T \to D$ and such an interval $J_n = (a_n, b_n)$, that the restriction φ_n on J_n is a solution of the differential equation

$$x' = h_n(t)\, x;$$

2) $\lim\limits_{n \to \infty} a_n = -\infty$ and $\lim\limits_{n \to \infty} b_n = +\infty$.

Then: the set $\{\varphi_n \mid n \in \mathbb{N}\}$ *is compact in the space* $(T; D)$ *and the limit of any convergent subsequence of the sequence* $\{\varphi_n\}$ *of points of the space* $(T; D)$ $(T; D)$ *is a solution of the differential equation*

$$x' = g(t)\, x.$$

Proof. 1. Let $l > 0$. Since

$$\sup_{|t| < l,\ x \in D} \| g(t)\, x \| < +\infty,$$

in view of (5) there will be found such $b > 0$, that

$$\sup_{|t| < l,\ x \in D} \| h_n(t)\, x \| \leqslant b$$

for any $n \in \mathrm{N}$. Choose $n_0 \in \mathrm{N}$ such that

$$[-l, l] \subseteq (a_n, b_n)$$

for any natural $n \geqslant n_0$. For these values of n and for any t_0 and t from $[-l, l]$ the relation is fulfilled

$$\| \varphi_n(t) - \varphi_n(t_0) \| \leqslant \left\| \int_{t_0}^{t} h_n(s) \varphi_n(s)\, ds \right\| \leqslant b\, |t - t_0|,$$

Figure 5: Figure 5

from which it follows that the set $\{\varphi_n \mid n \in \mathbb{N}\}$ is equicontinuous. Considering that D is a compact set, on the basis of the Arzelà-Ascoli theorem (see [15], pp. 30 — 33) we conclude that the set $\{\varphi_n \mid n \in \mathbb{N}\}$ is compact in $(T; D)$.

2. For simplification of notation, we choose the sequence $\{\varphi_n\}$ itself as a subsequence. Assume that it converges and let

$$\psi = \lim_{n\to\infty} \varphi_n. \tag{6}$$

Let us show that for any $t \in T$ the equality holds

$$\psi(t) = \psi(0) + \int_0^t g(s)\psi(s)\,ds. \tag{7}$$

For definiteness, we will assume $t > 0$. Let us choose $n_0 \in \mathbb{N}$ such that $[0, t] \subseteq (a_n, b_n)$ for all natural $n \geqslant n_0$. For these values of n, the following relation holds:

$$\| \psi(t) - \psi(0) - \int_0^t g(s)\psi(s)\,ds \| \leqslant \| \psi(t) - \varphi_n(t) \| + \| \varphi_n(0) - \psi(0) \| +$$

$$+ \int_0^t \| h_n(s)\varphi_n(s) - g(s)\varphi_n(s) \|\,ds + \int_0^t \| g(s)\varphi_n(s) - g(s)\psi(s) \|\,ds.$$

Hence, considering (5) and (6), as well as the fact that the mapping $(s, x) \to g(s)\,x$ of the product $[0, t] \times D$ into B is uniformly continuous, we become convinced of the validity of (7).

The lemma is proved.

Proof of Theorem 1. For definiteness, we will assume that the solution φ_0 is defined on $J = [t_0, +\infty)$.

First, we will show that there exists a fully recurrent solution of equation (3), which is an ω-limit image of the function φ_0.

Let us define the mapping $\varphi : T \to D$ by the relation

$$\varphi(t) = \begin{cases} \varphi_0(t) & \text{for } t > t_0; \\ \varphi_0(t_0) & \text{for } t \leqslant t_0. \end{cases}$$

It is clear that the set $\varphi(T)$ is compact in D, moreover, the function is uniformly continuous. Indeed, whatever the real t_1 and t_2 may be, the inequality holds

$$\| \varphi(t_1) - \varphi(t_2) \| \leqslant |t_1 - t_2| \sup_{t\in T} \| f(t)\varphi(t) \|.$$

At the same time

$$\sup_{t\in T} \| f(t)\varphi(t) \| < \infty,$$

since the function f is fully recurrent.

Let us consider the dynamic system $\sigma(T; D)$. From what has been proved, it follows that the motion σ_φ is Lagrange stable (see [4], Lemma 1.65). This means the set Ω_φ of all ω-limit points of the motion σ_φ contains some compact minimal set Ψ (see [2], p. 402, Corollary 2). Let us choose an arbitrary $\psi \in \Psi$. Note that the motion σ_ψ is recurrent (see [2], p. 402, Theorem 27). Since $\psi \in \Omega_\varphi$, there exists a sequence of positive numbers $\{\tau_n\}$, converging to $+\infty$, for which

$$\lim_{n\to\infty} \sigma_\varphi(\tau_n) = \psi. \tag{8}$$

Figure 6: Figure 6

According to (2.10) and (3.2),

$$\lambda^{k+1} = F((k-2)T, T) + 3[F(kT, T) - F((k-1)T, T)], \quad (4.4)$$

where $F(kT, T)$ $(k = 1, 2, \ldots, N)$ is calculated by the formula

$$F(kT, T) = \frac{1}{T}(x_2^k - x_2^{k-1}) - \frac{1}{6}(x_1^{k-1} + 4x_1^{k-\frac{1}{2}} + x_1^k) - u^k, \quad (4.5)$$

and $F(0, T)$, $F(-T, T)$, $F(-2T, T)$ are assumed to be equal to zero.

The equation for ontermining the control, obtained from condition (3.6) and cootrerst-crding to formylas (4.2), meıt the form

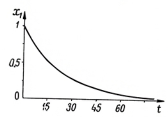

Fig. 1.

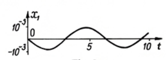

Fig. 2.

$$(\alpha + \alpha^2 + \beta^2) x_1^k + (\beta + 2\alpha\beta) x_2^k + (\alpha^2 + \beta^2)(u^{k+1} + \lambda^{k+1}) = 0.$$

Hence

$$u^{k+1} = -\lambda^{k+1} - \left(1 + \frac{\alpha}{\alpha^2 + \beta^2}\right) x_1^k - \frac{\beta(1 + 2\alpha)}{\alpha^2 + \beta^2} x_2^k. \quad (4.6)$$

Calculations were carried out on a digital computer with a step $T = 0.1$ for various disturbing functions. In all cases, the qauectative picture of the motions turned out to be the same. The results of the calculations for the case $g(t) = \sin t$ are presented graphically in Figures 1 and 2.

In Fig. 1 the change of the coordinate x_1 is shown for the initial data $x_1(0) = 1$, $x_2(0) = 0$; in Fig. 2 the change of the same coordinate is shown in the case of zero initial data shonra $x_1(0) = 0$, $x_2(0) = 0$.

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Figure 7: Figure 7

The validity of the formulated proposition is established on the basis of theorem 3, remark 4 and corollary 3.

In conclusion, we give one criterion for the existence of a unique fully recurrent solution of a linear nonhomogeneous system of differential equations.

Theorem 4. Let A be a real matrix of order n, the spectrum of which does not intersect with the imaginary asis. Whatever the fully recurrent mapping $a : T \to \mathrm{T}^n$, there exists a unique fully recurrent solution of the differential equation

$$x' = Ax + \alpha(t). \qquad (19)$$

Proof. Let a be a fully recurrent mapping $T \to \mathrm{T}^n$. Since every fully recurrent function is bounded, according to theorem 2.3 of paper [19], there is a unique Lagrange stable solution of equation (19), which, according to corollary 4, is the only fully recurrent solution of this equation.

Theorem 4 is analogous to the criterion for the existence of a unique almost periodic solution of equation (19), proved in [20].

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Figure 8: Figure 8

2. Proof of Boundedness of the Solution. Let

$$(x, f(x, y, u, t)) \leqslant C\left(|x|^2 + |y|^2 + 1\right),\ t \in T. \qquad \text{(Eq. 9)}$$

Let us construct the auxiliary function

$$X(t) = |x(t)|^2 + |y(t)|^2 + 1. \qquad \text{(Eq. 10)}$$

As before, we will use the notation $\tau = \tau(t) = t - h(x(t),\ u(t),\ t)$. Assuming that $u(t)$ is differentiable, from (Eq. 10) with regard to (Eq. 9) we have

$$\frac{dX(t)}{dt} = 2\left(x(t), f(x, y, u, t)\right) + 2\left(x(t), f(x, y, u, t)\right)\big|_{t=\tau(t)} \tau'(t) \leqslant$$
$$\leqslant 2C\left(X(t) + X[\tau(t)]\,\tau'(t)\right).$$

Integrating this differential inequality, we get

$$X(t) \leqslant X(t_0) + 2C\Big[\int_{t_0}^{t} X(t)\,ds + \int_{t_0}^{t} X[\tau(t)]\,\tau'(s)\,ds\Big] =$$
$$= X(t_0) + 2C\Big[\int_{t_0}^{t} X(s)\,ds + \int_{\tau(t_0)}^{\tau(t)} X(s)\,ds\Big] \leqslant X(t_0) + 2C\Big[\int_{t_0}^{t} X(s)\,ds +$$
$$+ \int_{\tau(t_0)}^{t_0} X(s)\,ds + \int_{t_0}^{t} X(s)\,ds\Big] \leqslant X(t_0) + 4C\int_{t_0}^{t} X(s)\,ds +$$
$$+ 2C(2K^2 + 1)\,h(\varphi(t_0), u(t_0), t_0) = a + 4C\int_{t_0}^{t} X(s)\,ds.$$

Here

$$a = |x(t_0)|^2 + |y(t_0)|^2 + 1 + 2C(2K^2 + 1)\,h_0,$$
$$K = \max_{t \in S_0} |\varphi(t)|, \quad h_0 = h(\varphi(t_0), u(t_0), t_0).$$

Hence

$$X(t) \leqslant a e^{4C(t - t_0)} \leqslant a e^{4C(t_1 - t_0)}, \quad t \in [t_0, t_1].$$

From the boundedness of $X(t)$ on $[t_0, t_1]$ it follows that

$$|x(t)| \leqslant K_1, \quad K_1^2 \leqslant a e^{4C(t_1 - t_0)} - 1.$$

3. Proof of Continuity of the Solution with Respect to Control. Let condition (3) be satisfied for admissible controls $u(t)$ and $\tilde{u}(t)$

$$\int_{t_0}^{t} \|\tilde{u}(\theta) - u(\theta)\|\,d\theta < \delta.$$

Let us estimate

$$\|\tilde{x}(t) - x(t)\| \leqslant \int_{t_0}^{t} \|f(\tilde{x}, \tilde{y}, \tilde{u}, \theta) - f(x, y, u, \theta)\|\,d\theta \leqslant$$
$$\leqslant L_1 \int_{t_0}^{t} \left[\|\tilde{x}(\theta) - x(\theta)\| + \|\tilde{y}(\theta) - y(\theta)\| + \|\tilde{u}(\theta) - u(\theta)\|\right] d\theta.$$

Figure 9: Figure 9

Consider separately

$$\|y(\theta)-y(0)\| = \|x(\theta-h[x(0),u(0),0])-x(0-h[x(0),u(0),0))\| = \\ = \|x(\tau)-x(\tau)\| \leqslant \|x(\tau)-x(\tau)\| + \|x(\tau)-x(\tau)\|.$$

Considering, that $\|x(\tau)-x(\tau)\| \equiv 0$ when $\tau \leq t_0$, we obtain

$$\int_{t_0}^{t} \|x(\tau)-x(\tau)\|\,d\theta = \int_{t_0}^{t} \|x(\theta-h(x,u,0)) - \\ - x(\theta-h(x,u,0))\|\,d\theta \leqslant \int_{t_0}^{t} \|x(\theta)-x(\tau)\|\,d\theta.$$

Further,

$$\|x(\tau)-x(\tau)\| \leqslant M|\tau-\tau| = M|h(x,u,0)-h(x,u,0)| \leqslant \\ \leqslant ML_2[\|x-x\| + \|u-u\|].$$

Finally

$$\|x(t)-x(t)\| \leqslant L_1(2+ML_2)\int_{t_0}^{t}\|x(\theta)-x(\theta)\|\,d\theta + \\ + L_1(1+ML_2)\int_{t_0}^{t}\|u(\theta)-u(0)\|\,d\theta < L_1(2+ML_2)\int_{t_0}^{t}\|x(\theta) - \\ - x(0)\|\,d\theta + L_1(1+ML_2)\delta.$$

Therefore [5]

$$\|x(t)-x(t)\| \leqslant L_1(1+ML_2)\,\delta e^{2c(2+ML_2)(t-t_0)} = \varepsilon, \quad t\in T.$$

4. Proof of continuity of the solution with initial function. Consider two solutions $x(t)$ and $x(t)$ to systems (1), generated by one and the same controlled $u(t)$, with given initial functions $\varphi(t)$ and $\varphi(t)$, corresponding solutions (4):

$$\|\varphi(t_0)-\varphi(t_0)\| < \delta. \quad \int_{t_0}^{t}\|\varphi(\theta)-\varphi(0)\|d\theta < \delta.$$

Let us estimate the difference

$$\|x(t)-x(t)\| \leq \|\varphi(t_0)-\varphi(t_0)\| + \int_{t_0}^{t}\|f(x,y,u,0)-f(x,y,u,0)\|d\theta \leq \\ \leqslant \|\varphi(t_0)-\varphi(t_0)\| + L_1\int_{t_0}^{t}[\|x(0)-x(0)\| + \|y(0)-y(0)\|]d\theta.$$

There

$$\|y(0)-y(0)\| = \|x(\tau)-x(t)\| \leq \|x(\tau)-x(\tau)\| + \|x(\tau)-x(t)\| \leq \\ \leqslant \|x(\tau)-x(\tau)\| + M|h(x,u,0)-h(x,u,0)\| \leq \\ \leqslant \|x(\tau)-x(\tau)\| + ML_2\|x(0)-x(0)\|.$$

The inequality holds

Figure 10: Figure 10

$$\int_{t_0}^{t} \|\bar{x}(\tilde{\tau}) - x(\tilde{\tau})\| \, d\theta \leqslant \int_{s_0} \|\bar{\varphi}(0) - \varphi(0)\| \, d\theta + \int_{s_0}^{t} \|\bar{x}(0) - x(0)\| \, d\theta.$$

Therefore

$$\|\bar{x}(t) - x(t)\| \leqslant \|\bar{q}(q) - \varphi(q)\| + L_1(2 + ML_2) \int_{t_0}^{t} \|\bar{x}(0) - x(0)\| \, d\theta +$$

$$+ L_1 \int_{t_0} \|\bar{q}(0) - \varphi(0)\| \, d\theta < L_1(2 + ML_2) \int_{t_0}^{t} \|\bar{x}(0) - x(0)\| \, d\theta + (L_1 + 1)\,\delta.$$

From the last inequality it follows that

$$\|\bar{x}(t) - x(0)\| < (L_1 + 1)\,\delta e^{\delta_t(2+ML_2)(t-t_0)} = \varepsilon, \quad t > t_0.$$

5. Proof of Theorem 2. Let $Q(x_t(\cdot), t)$ — be some set of elements $q(x_t(\cdot), t)$, where $x_t(\cdot) = x(\tau)$, $\tau \in [t_0, t]$, — are arbitrary continuous functions on $\mathcal{T}$. We call the set $Q(x_t(\cdot), t)$ *upper semicontinuous with respect to inclusion* if for any t and $x_t(\cdot)$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|t' - t| < \delta$ and there $|t' - t| < \delta$ is

$$\|x'_{t'}(\cdot) - x_t(\cdot)\| = \max_{0 \in (t_0, t'); |[t_0, t]} \|x'(0) - x(0)\| < \delta$$

the following inclusion holds

$$Q(x'_{t'}(\cdot), t') \subseteq [Q(x_t(\cdot), t)]_\varepsilon.$$

$[QL_\varepsilon$ — ε-neighborhood of set Q.

First, let's prove the following lemma.

Lemma 1. *If the functions $f(x, y, u, t)$, $h(x, u, t)$ and $\varphi(t)$ satisfy the conditions of Theorem 1 and, in addition, $\left|\frac{\partial h}{\partial t}\right| \leq L_2$, then the set $R(x_t(\cdot), t)$ is upper semicontinuous with respect to inclusion.*

Proof. Assume the contrary: the set $R(x_t(\cdot), t)$ does not possess the property of semicontinuity, i.e., for given t and $x_t(\cdot)$ there exists $\varepsilon > 0$ such that no matter how small $\delta > 0$ is, the set seat $\delta > 0$ is, the set $R(x'_{t'}(\cdot), t') \subseteq [R(x_t(\cdot), t)]_\varepsilon$, for $|t' - t| < \delta$ and $\|x'_t(\cdot) - x_t(\cdot)\| < \delta$. This means that there exists $u \in U$ such that

$$\|f(x'(t'), x'(t' - h(x'(t'), u, t')), u, t') -$$

$$- f(x(t), x(t - h(x(t), u, t]), u, t)\| > \varepsilon \qquad \text{(n. 11)}$$

where

$$|t' - t| < \delta \quad \|x'_t(\cdot) - x_t(\cdot)\| = \max_{0 \in (t_0, t'), 1[t_0, t]} \|x'(0) - x(0)\| < \delta. \quad \text{(n. 12)}$$

where δ is arbitrarily small. Let's show that conditions (11), (12) contradict the continuity of the function $f(x, y, u, t)$ on the set P.

Consider two points t', $t \in \mathcal{T}$ (for definiteness, assume $t' > t$) and two functions $x'_{t'}(\cdot)$ and $x_t(\cdot)$ satisfying condition (12). Let's estimate

$$\|x' - x\| = \|x'(t') - x(t)\| \leqslant \|x'(t') - x'(t)\| + \|x'(t) - x(t)\| \leqslant$$

$$\leqslant M(t' - t) + \|x'(t) - x(t)\| < (M + 1)\delta.$$

Figure 11: Figure 11

$$\|y' - y\| = \|x'(t' - h[x'(t'), u, t']) - x(t - h[x(t), u, t])\| =$$
$$= \|x'(\tau') - x(\tau)\| \leqslant \|x'(\tau') - x'(\tau)\| + \|x'(\tau) - x(\tau)\|.$$

Here

$$\|x'(\tau') - x'(\tau)\| \leqslant M|\tau' - \tau| \leqslant M\left[(t' - t) + |h(x'(t'), u, t') -\right.$$
$$\left. - h(x(t), u, t)|\right] < M\left[\delta + L_2(\|x'(t') - x(t)\| + (t' - t))\right] <$$
$$< M\left[\delta + L_2((M+1)\delta + \delta)\right] = M\delta(1 + L_2M + 2L_2).$$

Therefore

$$\|y' - y\| < M\delta(1 + L_2M + 2L_2) + \delta.$$

Thus, we have shown that in the set P there exist two points (x, y, u, t) and (x', y', u, t') such that for

$$|t' - t| < \delta, \quad \|x' - x\| < (M+1)\delta, \quad \|y' - y\| < [M(1 + L_2M + 2L_2) + 1]\delta$$

the corresponding difference

$$\|f(x', y', u, t') - f(x, y, u, t)\| > \epsilon$$

for all arbitrarilly small δ. But this is impossible, since $f(x, y, u, t)$ is continuous on the set P.

Lemma proved.

Let us proceed to the proof of theorem 2.

Let $\bar{u}(t)$ be some admissible control, and $\bar{x}(t)$ be the corresponding solution of the system (1), (2). The value of the functional $J(\bar{u}) = \Phi(\bar{x}(t_1))$ will be donated by a.

Let us consider the set of solutions $\{x(t), t \in T : \Phi(x(t_1)) \leq a\}$ with the initial condition (2) and $u \in U$. This net is not empty. If it is finive, then the theorem is proved.

Let it be inconsive. Let us chinose from it a secledeance Γ such $x(t)$, $t \in \Gamma$, for which the seclegance of cootrersecting values $\Phi(x(t_1))$ convegates to anome number $\bar{a} = \text{lnf}\{\Phi(x(t_1))\}$. The limit $\bar{a}$ is ewists, since the secledataretensmets $\{\Phi(x(t_1))\}$ is bounded and monotone.

Mnowectro Γ is uniformlly bounded ($\|x(t)\| \leq K_1$, $t \in T$, for all $x(t) \in \Gamma$) and uniformly continuous, since for all $x(t) \in T$ the derivati-aits are $\dfrac{dx(t)}{dt}$ organeved by one and the same number:

$$\left|\frac{dx(t)}{dt}\right| = |f(z, u, u, t)| \leqslant M \text{ almost everywhere on } T.$$

Therefore, from the sequence Γ one can select a subsequence $x^{(i)}(t)$, $i = 1, 2, \ldots$, uniformly converging to some function $\bar{z}(t)$. Let us show valuces that the limit function $\bar{x}(t)$ is the optimal trajectory of the system (1), (2), (5).

Let $dx^{(i)}(t)$ $\dfrac{dx^{(i)}(t)}{dt} = p_i(t)$ and $\dfrac{d\bar{x}(t)}{dt} = \bar{p}(t)$. Due to the uniform conno-periestnosity of functions $x^{(i)}(t)$, $i = 1, 2, \ldots$, and condition $x^{(i)}(\tau) = \varphi(\tau)$, $\tau \in S_0$, for the limineit function $\bar{x}(t)$ we have

$$\bar{x}(t) = \varphi(t), \quad t \in S_0, \quad \Phi(\bar{x}(t_1)) = \bar{a}.$$

Figure 12: Figure 12

Since all $x^{(i)}(t)$, $i = 1, 2, \ldots$, are absolutely continuous and $\left|\frac{dx^{(i)}(t)}{dt}\right| \leqslant M$, then $\bar{x}(t)$ is also absolutely continuous, the function $p(t) = \frac{d\bar{x}(t)}{dt}$ is defined almost everywhere on T and at points of existence of the derivative $\left|\frac{dx(t)}{dt}\right| \leqslant M$.

Let at the point t^*, $t^* \in T$, the derivative $\frac{d\bar{x}(t^*)}{dt}$ exists exist. We will show then that for any arbitrarily small $\varepsilon > 0$ there exists a $\delta > 0$ such that for $|t - t^*| < \delta$

$$\frac{\bar{x}(t) - \bar{x}(t^*)}{t - t^*} \in [R(\bar{x}_{t^*}(\cdot), t^*)]_\varepsilon.$$

By construction of the function $\bar{x}(t)$ we have:

$$\frac{\bar{x}(t) - \bar{x}(t^*)}{t - t^*} = \lim_{i\to\infty} \frac{x^{(i)}(t) - x^{(i)}(t^*)}{t - t^*}.$$

But

$$x^{(i)}(t) = x^{(i)}(t^*) + \int_{t^*}^{t} p_i(t)\,dt,$$

therefore

$$\frac{\bar{x}(t) - \bar{x}(t^*)}{t - t^*} = \frac{1}{t - t^*} \lim_{i\to\infty} \int_{t^*}^{t} p_i(\tau)\,d\tau = \lim_{i\to\infty} \int_0^1 p_i(t^* - (t - t^*)s)\,ds.$$

Functions

$$p_i(\tau) = p_i(t^* + (t - t^*)s) \in R\left(x_\tau^{(i)}(\cdot), \tau\right)$$

for all $s \in [0, 1]$. It is easy to show that for sufficiently large i and $|\tau - t^*| < \delta$

$$R\left(x_\tau^{(i)}(\cdot), \tau\right) \subset [R(\bar{x}_{t^*}(\cdot), t^*)]_\varepsilon. \qquad \text{(p. 13)}$$

Indeed, since $\bar{x}(t) = \lim\limits_{i\to\infty} \lim\limits_{i\to\infty} x^{(i)}(t)$, then for a given $\delta > 0$ one can indicate an N such that for $i \geqslant N$

$$|\bar{x}(t) - x^{(i)}(t)| < \delta$$

for all $t \in [t_0, t_1]$. Therefore for the indicated i

$$\max_{\theta \in [t_0, t_1] \cap [t^*, \tau]} |\bar{x}(\theta) - x^{(i)}(\theta)| < \delta. \qquad \text{(p. 14)}$$

From (p. 14) it follows (p. 13), if $|\tau - t^*| < \delta$ and $i \geqslant N$. Then under these same conditions

$$p_i(t^* - (t - t^*)s) \in [R(\bar{x}_{t^*}(\cdot), t^*)]_\varepsilon \text{ for all } s \in [0, 1]. \qquad \text{(p. 15)}$$

According to the condition of the theorem the set $R(\bar{x}_{t^*}(\cdot), t^*)$ (andy, also $[R(\bar{x}_{t^*}(\cdot), t^*)]_\varepsilon$) is convex. Therefore from (p. 15) it follows that

$$\int_0^1 p_i(t^* - (t - t^*)s)\,ds \in [R(\bar{x}_{t^*}(\cdot), t^*)]_\varepsilon.$$

Figure 13: Figure 13

and

$$\frac{x(t) - x(t^*)}{t - t^*} \in [R(x_{t^*}(\cdot), t^*)]_\varepsilon. \qquad \text{(eq. 16)}$$

Further, for sufficiently small δ, the inequality is satisfied simultaneously with (eq. 16)

$$\left| \frac{dx(t^*)}{dt} - \frac{x(t) - x(t^*)}{t - t^*} \right| < \varepsilon,$$

from which it follows that

$$\frac{dx(t^*)}{dt} \in [R(x_{t^*}(\cdot), t^*)]_{2\varepsilon}.$$

Since the number ε can be arbitsrarily small, and the set $R(x_{t^*}(\cdot), t^*)$ $t^*)$ is closed, then $\frac{dx(t^*)}{dt} \in R(x_{t^*}(\cdot), t^*)$. This means that there exists a $u(t) \in U$, such that

$$\frac{dx(t^*)}{dt} = f(x(t^*), x(t^* - h[x(t^*), u(t^*), t^*]), u(t^*), t^*).$$

Thus, for almost all $t^* \in T$, there is a $u(t^*) \in U$, substerfying the last equastity. Following [3], it is not difficult to prove that $u(t^*)$ can be unöoan so, that the wrone function $u(t^*)$, $t^* \in T$, is unmerparable.

Theorem 2 is proven.

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Figure 14: Figure 14