ON EXTREMALLY DISCONNECTED BICOMPACTA

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.83+519.5

MATHEMATICS

B. EFIMOV

ON EXTREMALLY DISCONNECTED BICOMPACTA

(Presented by Academician P. S. Aleksandrov, 5 IV 1966)

P. S. Aleksandrov, at a seminar at Moscow University, posed the following problem: does there exist a universal extremally disconnected bicompactum of weight τ*, topologically containing every other extremally disconnected bicompactum of weight $\leqslant \tau$? In this note it is proved (Theorem 1) that for all admissible cardinal numbers such a bicompactum exists. Here the cardinal number τ is called admissible if $\tau^{\aleph_0} = \tau$. It is further proved (Theorem 2) that this bicompactum is homogeneous with respect to character**. This answers a question proposed by M. Ya. Antonovskii: do there exist extremally disconnected bicompacta homogeneous with respect to character? Finally, Theorem 3 asserts that the weight of any infinite extremally disconnected bicompactum is an admissible cardinal number, which completely solves both problems.

§ 1. V. I. Ponomarev ([1–3]) and A. Gleason ([4]) proved that for every completely regular (Gleason—for locally bicompact) space X there exists an irreducible perfect*** extremally disconnected preimage pX, which the former called an absolute and the latter a projective space. For a bicompact space X, the simplest construction is that of S. Iliadis ([5,6]), S. V. Fomin ([7]), which consists in the following. The points of pX are declared to be all possible maximal centered systems $\mathfrak{F}$ consisting of open subsets of X. As a base of closed sets in pX one takes sets of the form $\Gamma_U = \mathcal{E}(\mathfrak{F} : U \in \mathfrak{F})$, if U is open in X. It can be proved that the space pX is bicompact and extremally disconnected. Assigning to each point $x \in pX$ (= a maximal centered system $\mathfrak{F}$) its unique point of contact from X (X a bicompactum), we obtain an irreducible mapping of pX onto X.

Lemma 1. *The weight of the absolute pX of a bicompactum X is equal to the cardinality of all canonically closed subsets of X**.*

Proof. Let $f : pX \to X$ be an irreducible mapping of pX onto X. To each canonically closed $[U] \subset X$ we assign

$$V = [f^{-1}U]_{pX},$$

a certain open-and-closed subset of pX. It can be proved that, by virtue of the irreducibility and perfection of the mapping f, this rule establishes a one-to-one correspondence between all open-and-closed subsets of pX and all canonically closed subsets of X. Since the cardinality of any open-and-closed base of pX is equal to the weight of pX, the lemma follows.

* A topological space X is called **extremally disconnected** if the closure $[U]_X$ of any open $U \subset X$ is again open. The minimum of the cardinalities of open bases of the space X is called the **weight** and is denoted by wX.

** The **character** $\chi(F, X)$ of a set F in X is the minimum of the cardinalities of fundamental systems of neighborhoods of F in X.

*** A continuous mapping $f : X \to Y$ is **irreducible** if there is no proper closed subset $F \subset X$ for which $f(F) = Y$. A mapping is called **perfect** if it is closed and bicompact.

**** A set $F = [U]$ is called **canonically closed** if it is the closure of a nonempty open U.

Definition. A cardinal number τ will be called **admissible** if $\tau^{\aleph_0} = \tau$.

Theorem 1. *Let τ be an admissible cardinal number, $D^\tau = \prod D^{0,1}_\alpha$, $\alpha \in A$, $|A| = \tau^*$, be the generalized Cantor discontinuum of weight τ. Then the absolute pD^τ is a universal extremally disconnected bicompactum of weight τ.*

Proof. First of all, let us note that pD^τ is an extremally disconnected bicompactum of weight τ. Indeed, in the space D^τ every canonical closed set has type G_δ (Theorem 5 ([8])). By virtue of bicompactness and zero-dimensionality of D^τ, every closed set of type G_δ in D^τ is the intersection of a countable number of open-and-closed sets. Since the cardinality of all open-and-closed sets in D^τ is τ, the cardinality of all closed sets of type G_δ in D^τ is $\tau^{\aleph_0} = \tau$ (τ is admissible!). Applying Lemma 1, we obtain that $w(pD^\tau) = \tau$. Let us prove the universality of pD^τ. Let X be an arbitrary extremally disconnected bicompactum of weight $\leq \tau$. By a theorem of N. B. Vedenisov ([9]), the bicompactum X, as a zero-dimensional space in the sense of ind, can be topologically embedded in D^τ. Let $X \subset D^\tau$ and let $f : pD^\tau \to D^\tau$ be irreducible. Then $Y = f^{-1}X \subset pD^\tau$. By Brawer' s theorem ([10], p. 27), in pD^τ there exists $X^* \subset Y \subset pD^\tau$ such that $f(X^*) = X$, and on X^* the mapping f is irreducible. This means, by virtue of the extremal disconnectedness of X, that X^* is homeomorphic to X (Lemma 2.3 ([4])), as was required to prove.

§ 2. A system $\mathfrak{B} = \{[U]\}$ consisting of nonempty canonical closed sets will be called a **δ-system of the closed set** $F \subset R$, if: 1) for any two $[U]$, $[V] \in \mathfrak{B}$ we have $[U] \cap [V] = [W] \in \mathfrak{B}$, and 2) $\cap [U] = F$ for all $[U] \in \mathfrak{B}$. The minimum of the cardinalities of all δ-systems of the closed set $F \subset R$ will be called the **δ-character** of F in R, and will be denoted by $\delta(F, R)$. If R is a compact metric space, then $\delta(F, R) \leq \psi(F, R) = \chi(F, R)^{**}$. In general, only the inequality holds

$$\delta(F, R) \leq \psi(F, R) \leq \chi(F, R).$$

a) Let $f : X \to Y$ be an irreducible mapping of the bicompactum X onto Y. Then $\delta(F, X) \geq \delta(f, F, Y)$.

This proposition follows from the fact that the irreducible image of a δ-system F is a δ-system fF.

b) For all $x \in D^\tau$, we have $\delta(x, D^\tau) = \psi(x, D^\tau) = \chi(x, D^\tau)$.

Let us prove b). Since $\delta(x, D^\tau) \leq \psi(x, D^\tau) = \chi(x, D^\tau) = \tau$ ([8]), it is enough to prove that $\tau \leq \delta(x, D^\tau)$. Let $\mathfrak{B} = \{[U]\}$ be a minimal δ-system of the point $x \in D^\tau$. Since in D^τ every canonical closed set has type G_δ (Theorem 5 ([8])), for every $[U] \in \mathfrak{B}$ we have $[U] = \bigcap_{k=1}^{\infty} O_k[U]$; hence

$$x = \bigcap [u] = \bigcap \bigcap_{k=1}^{\infty} O_k[U],$$

where the first intersection is taken over all $[U] \in \mathfrak{B}$. Thus we obtain that $\tau = \psi(x, D^\tau) \leq \aleph_0 |\mathfrak{B}| = |\mathfrak{B}|$, as was required to prove.

Theorem 2. *Let τ be an admissible cardinal number. Then pD^τ is an extremally disconnected bicompactum homogeneous with respect to character τ**.*

Proof. Since τ is an admissible cardinal number, by Theorem 1 we have $\chi(x, pD^\tau) \leq w(pD^\tau) = \tau$. Therefore it is enough to prove that $\chi(x, pD^\tau) \geq \tau$. Suppose that $\chi(x, pD^\tau) = \mathfrak{m} < \tau$. Let $f : pD^\tau \to D^\tau$ be irreducible; then, by property a), we have $\mathfrak{m} = \chi(x, pD^\tau) \geq \delta(x, pD^\tau) \geq \delta(fx, D^\tau)$. By property b), $\delta(fx, D^\tau) = \chi(fx, D^\tau) = \tau$. Thus–

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* $|A|$ is the cardinality of the set A.

** The **pseudocharacter** $\psi(F, X)$ of the set F in X is the least cardinal number $\mathfrak{m}$ such that $F = \bigcap O_\alpha$, $\alpha \in A$, all O_α are open and $|A| = \mathfrak{m}$. If F is closed and X is a bicompactum, then $\chi(F, X) = \psi(F, X)$.

*** A space X is called **homogeneous with respect to character** τ if $\chi(x, X) = \tau$ for all $x \in X$.

Thus we have obtained that $\mathfrak{m} \geq \tau$, contrary to the assumption. The theorem is proved.

§ 3. The sign $\oplus$ denotes the disjoint union of spaces X_k; βX denotes the Stone–Čech compactification of X. Let

$$X = \bigoplus_{k=1}^{\infty} X_k$$

and $f_k : X_k \to Y_k$; then by

$$f = \bigoplus_{k=1}^{\infty} f_k$$

we denote the mapping

$$f : X \to \bigoplus_{k=1}^{\infty} Y_k,$$

for which the restriction $f/X_k = f_k$.

Lemma 2. Let $\{\mathfrak{m}_k\}$ be a countable set of infinite cardinal numbers and

$$\tau = \prod_{k=1}^{\infty} \mathfrak{m}_k.$$

Then $\tau^{\aleph_0} = \tau$.

Proof. It is enough to prove that $\tau^{\aleph_0} \leq \tau$. Since

$$\mathfrak{m}_k \leq \sum_{k=1}^{\infty} \mathfrak{m}_k,$$

we have

$$\tau = \prod_{k=1}^{\infty} \mathfrak{m}_k \leq \prod \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right) = \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0}.$$

Therefore, using the distributivity of product with respect to sum ((12), p. 191), we obtain

$$\tau^{\aleph_0} \leq \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0 \cdot \aleph_0} = \left(\sum_{k=1}^{\infty} \mathfrak{m}_k \right)^{\aleph_0} = \sum_{\alpha \in A} \prod_{i=1}^{\infty} \mathfrak{m}_{k(i)}, \quad \alpha \leq |A| \prod_{k=1}^{\infty} \mathfrak{m}_k =$$

$$= \aleph_0^{\aleph_0} \prod_{k=1}^{\infty} \mathfrak{m}_k = \prod_{k=1}^{\infty} (\aleph_0 \cdot \mathfrak{m}_k) = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau.$$

The lemma is proved.

Lemma 3. Let

$$X = \bigoplus_{k=1}^{\infty} X_k,$$

where the X_k are bicompacta and $wX_k \geq \aleph_0$. Then

$$w(\beta X) = \prod_{k=1}^{\infty} wX_k.$$

Proof. Put $wX_k = \mathfrak{m}_k$ and

$$\prod_{k=1}^{\infty} \mathfrak{m}_k = \tau.$$

We first prove that $w(\beta X) \leq \tau$. For this it is enough to prove that the cardinality of the set of all bounded continuous functions on X does not exceed τ. Denote this set by $\mathfrak{N} = \{f\}$. Let $f_k = f/X_k$ be the restriction of an arbitrary function $f \in \mathfrak{N}$ to X_k; then

$$f = \bigoplus_{k=1}^{\infty} f_k.$$

Conversely, a combination of arbitrary continuous functions f_k on X_k gives some continuous function on X. Hence

$$|\mathfrak{N}| \leq \prod_{k=1}^{\infty} |C(X_k)|,$$

if $C(X_k)$ is the ring of all continuous real-valued functions on X_k. Since

$$wC(X_k) = wX_k$$

(11) and $C(X_k)$ is a metric space, it follows that

$$|C(X_k)| \leq (wX_k)^{\aleph_0} = \mathfrak{m}_k^{\aleph_0}.$$

Further, applying Lemma 1, we obtain

$$|\mathfrak{N}| \leq \prod_{k=1}^{\infty} \mathfrak{m}_k^{\aleph_0} = \left(\prod_{k=1}^{\infty} \mathfrak{m}_k\right)^{\aleph_0} = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau.$$

Thus the inequality $w(\beta X) \le \tau$ is proved. We now prove that $w(\beta X) \ge \tau$. Since $wC(\beta X) = w(\beta X)$, to prove this inequality it is enough to find in $C(\beta X)$ a family $\mathfrak{M} = \{f\}$, $|\mathfrak{M}| \ge \tau$, consisting of functions whose pairwise distances are ≥ 1. Since $wC(X_k) = wX_k = \mathfrak{m}_k$, in $C(X_k)$ there exists a family $\mathfrak{M}_k = \{f_k\}$ of functions such that*: 1) $\rho(f_k, g_k) = \sup_{x\in X_k} |f_k(x) - g_k(x)| = 1,\ f_k \ne g_k \in \mathfrak{M}_k$; 2) $\sup_{x\in X_k} |f_k(x)| = 1,\ f_k \in \mathfrak{M}_k$; 3) $|\mathfrak{M}_k| = \mathfrak{m}_k$.

* If X_k is zero-dimensional, then for $\mathfrak{M}_k$ one may take the family of functions f for which $f(V) = 0$ and $f(X_k \setminus V) = 1$, where V is an arbitrary open-and-closed subset of X_k.

Next put $f = \bigoplus_{k=1}^{\infty} f_k$, if $f_k \in \mathfrak{M}_k$. Let f^* be the Stone–Čech extension of the function f to βX, which exists since f is bounded. Then the family $\mathfrak{M} = \{f^*\}$, which is obtained from $\{f\}$ if the f_k independently range over the $\mathfrak{M}_k$, is the desired one. Indeed,

$$|\mathfrak{M}| = \prod_{k=1}^{\infty} |\mathfrak{M}_k| = \prod_{k=1}^{\infty} \mathfrak{m}_k = \tau;$$

further, for any distinct $f^*, g^* \in \mathfrak{M}$ we have

$$\rho(f^*, g^*) = \rho(f, g) = \sup_k \sup_{x\in X_k} |f_k(x) - g_k(x)| = 1,$$

since for some X_k necessarily $f_k \ne g_k$. The lemma is proved.

Theorem 3. *The weight of an infinite extremally disconnected bicompactum is an admissible cardinal number.*

Proof. Let R be an extremally disconnected bicompactum, $|R| \ge \aleph_0$, and $wR = \tau$. Construct open-and-closed sets $U_k \subset R$ such that

$$U_k \cap U_n = \emptyset, \quad \text{if } k \ne n, \quad w(U_k) \ge \aleph_0, \quad \text{and} \quad \left[\bigcup_{k=1}^{\infty} U_k\right] = R.$$

Let T be the set of all isolated points in R^*. Consider two cases: 1) $[T] = R$ and 2) $[T] \ne R$. In the first case $|T| \ge \aleph_0$, hence $T = \bigoplus_{n=1}^{\infty} A_n$, $A_n \subset T$ and $|A_n| \ge \aleph_0$. Then $[A_n]_R = U_n$ are the required sets. In the second case the sets $[T]$ and $R \setminus [T]$ are open-and-closed, and $F = R \setminus [T]$ is extremally disconnected and contains no isolated points. By transfinite induction one can construct open-and-closed V_α, $\alpha \in A$, $|A| \ge \aleph_0$, such that $V_\alpha \subset F$, $V_\alpha \cap V_\beta = \emptyset$, if $\alpha \ne \beta$, $wV_\alpha > \aleph_0$, and

$$\left[\bigcup_{\alpha\in A} V_\alpha\right] = F.$$

Represent $A = \bigoplus_{n=1}^{\infty} A_n$, if $A_n \ne \emptyset$, $A_n \cap A_k = \emptyset$. In this case

$$U_n = \left[\bigcup_{\alpha\in A_n} V_\alpha\right]$$

are the required sets**. Thus, in both cases we have constructed an extremally disconnected space

$$X = \bigoplus_{n=1}^{\infty} U_k, \quad wU_k \geq \aleph_0,$$

and the U_k are bicompacta, which is dense in R. Consequently, by Gleason' s theorem (Theorem 4.1 ([4])), $R = \beta X$. Applying Lemma 3, we obtain

$$\tau = w(\beta X) = \prod_{k=1}^{\infty} wU_k,$$

and by Lemma 2, $\tau^{\aleph_0} = \tau$, which completely proves the theorem.

Corollary. *Let*

$$\tau = \sum_{k=1}^{\infty} \mathfrak{m}_k, \quad \mathfrak{m}_k \geq \aleph_0 \quad \text{and} \quad \mathfrak{m}_k < \tau.$$

Then there does not exist an extremally disconnected bicompactum of weight τ.

By virtue of the fact that an infinite extremally disconnected bicompactum is the Stone space of representation of some complete Boolean algebra and conversely, we obtain.

Theorem 4. *The cardinality of every complete infinite Boolean algebra is an admissible cardinal number.*

The author expresses gratitude to P. S. Aleksandrov for his attention to this work.

Moscow Geological Prospecting Institute
named after S. Ordzhonikidze

Received
31 III 1966

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* The set T may also be empty.

** If $|T| \geq \aleph_0$, then set $[T] = U_0$; if $|T| < \aleph_0$, then this cardinality has no effect on R.

Note: Figure translations are in progress. See original paper for figures.