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Abstract

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MATHEMATICAL PHYSICS

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ON RECTILINEAR MOTIONS OF AN IDEALLY PLASTIC MEDIUM

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This article studies the qualitative structure of stationary and nonstationary motions of an ideally plastic medium. The results obtained are presented by considering two specific problems; however, all the qualitative features noted are also characteristic of the general case of motion of an ideally plastic medium.

1. A variational principle for slow stationary motions of an ideally plastic medium was first formulated in the work of A. M. Markov ⁽¹⁾. For the case of rectilinear stationary motions this principle may be formulated as follows: the true motion of an ideally plastic medium differs from all its kinematically possible motions, with unchanged energy flux through the boundary, in that it minimizes the functional

$$I(u) = \int_{\omega} \tau_0 |\nabla u| d\omega - \int_{\Gamma} Qu d\Gamma - \int_{\omega} Xu d\omega, \quad (1)$$

where τ_0 is the yield limit of the medium; $u(x, y)$ is the velocity of motion of the particles; ω is the region occupied by the medium, with boundary Γ ; Q denotes external surface forces and X external body forces acting on the medium.

The functional space $W_1^{(1)}(\omega)$ corresponding to (1) is not reflexive. Therefore, generally speaking, there may be no minimizing function belonging to this space, and, since the functional $I(u)$ is not strictly convex, in the general case one cannot assert that the minimizing function will be unique, even if it exists. Such a structure of the functional $I(u)$ is connected with a number of peculiarities in the behavior of an ideally plastic medium.

Let us consider the motion of an ideally plastic medium in a pipe with cross-section ω under the action of a prescribed constant pressure gradient C . For a viscoplastic medium the corresponding problem was studied in ⁽²⁾. The functional $I(u)$ is rewritten in the form

$$I(u) = \int_{\omega} \tau_0 |\nabla u| d\omega - \int_{\omega} Cu d\omega, \quad (2)$$

Fig. 1: schematic of three regions $\omega_1, \omega_3, \omega_2$ with side a and separation d .

Figure 1: Fig. 1: schematic of three regions $\omega_1, \omega_3, \omega_2$ with side a and separation d .

where $u(x, y)$ is the velocity of particles of the medium in the direction of the pipe axis. We shall assume that on the boundary ω the no-slip conditions $u|_{\Gamma} = 0$ are satisfied.

It can be shown that if $C < C^*$, where C^* is the critical pressure gradient introduced in (2), then the minimizing function $u(x, y)$ exists, is unique, and $u \equiv 0$. Conversely, if $C > C^*$, then the minimizing function for (1) does not exist and $\inf I(u) = -\infty$. From the physical point of view this means that stationary motions cannot exist for a constant $C > C^*$. Finally, if $C = C^*$, then there exist minimizing sequences for (2) that converge to various functions which, generally speaking, do not belong to the space $W_1^{(1)}(\omega)$. We shall henceforth call these functions minimizing. They can be characterized as follows. Consider subdomains $\omega_1, \dots, \omega_s$

regions ω , bounded by the shear contours introduced in (2). Let $\chi(\omega_i)$ be the characteristic function of the region ω_i ($i = 1, \dots, s$). The class of minimizing functions coincides with the set of all linear combinations of characteristic functions with positive coefficients.

The last assertion shows that, under the same external conditions, the model of an ideally plastic medium admits a set of stationary motions that differ from one another not only in the magnitudes of the flow velocities, but also in configuration. As was shown in (2), this nonuniqueness cannot be eliminated by introducing a vanishingly small viscosity.

Let us now consider the problem of the motion of an ideally plastic medium caused by the displacement of a system of cylinders immersed in the medium. We shall assume that the axes of the cylinders are parallel and that the motion of the cylinders takes place in the direction of their axes. In this case (1) will have the form

$$I(u) = \int_{\Omega} \tau_0 |\nabla u| d\omega - \sum_{i=1}^N R_i u|_{\Gamma_i}, \quad (3)$$

where $\Omega = \omega \setminus \bigcup_1^N \omega_i$ is the region filled with the ideally plastic medium; ω_i is the cross section of the i -th cylinder; Γ_i is the boundary of ω_i ; R_i is the external force acting on the i -th cylinder. On the boundary Γ of the region ω and on Γ_i , sticking conditions are assumed to hold: $u|_{\Gamma} = 0$, $u|_{\Gamma_i} = a_i$. The constants a_i are not known in advance and must be determined in the process of solving the problem.

Fig. 1

Consider the contours $L_{i_1 \dots i_k}$ ($1 \leq k \leq N$) introduced in (2), and let $\text{mes } L_{i_1 \dots i_k}$ be the lengths of these contours. Then, if the inequality

$$\sum_{p=1}^k R_{i_p} < \tau_0 \text{mes } L_{i_1 \dots i_k}$$

holds, the minimizing function $u(x, y)$ exists, is unique, and $u \equiv 0$. Conversely, if the reverse inequality holds for at least one contour $L_{i_1 \dots i_k}$, then no minimizing function exists and $\inf I(u) = -\infty$. Finally, suppose that

$$\sum_{p=1}^k R_{i_p} \leq \tau_0 \text{mes } L_{i_1 \dots i_k}$$

for all sets of indices $i_1 \dots i_k$, and that equality holds for several contours $L_{j_1 \dots j_k}$. Denote by $\chi_{j_1 \dots j_k}$ the characteristic function of the region bounded by the contour $L_{j_1 \dots j_k}$. In this case one can construct a sequence of functions minimizing the functional (3) that converges to an arbitrary linear combination of the characteristic functions $\chi_{j_1 \dots j_k}$ with positive coefficients.

As in the preceding problem, here there is a set of different stationary states, differing in the magnitudes of the velocities and in their configurations.

Let us illustrate these statements on a concrete example. Let $N = 2$, and let ω_1 and ω_2 be squares of side a (Fig. 1). If $d > a$, $R_1 < 4\tau_0 a$, $R_2 = 4\tau_0 a$, then there is a motion of the second cylinder with arbitrary constant velocity; the whole medium and the first cylinder are then at rest. Further, if $d < a$, $R_1 + R_2 = 2\tau_0(3a + d)$, then the cylinders and the part of the medium ω_3 enclosed between them move as a single whole with arbitrary constant velocity. If, finally, $d = a$ and $R_1 + R_2 = 8\tau_0 a$, then it is possible

the following types of stationary motions: cylinder ω_1 moves with velocity v_1 , cylinder ω_2 moves with velocity v_2 , and part of the medium ω_3 with velocity v_3 ; v_1, v_2, v_3 are arbitrary positive numbers.

2. Let us now proceed to the study of nonstationary rectilinear motions of an ideally plastic medium. We formulate the following variational principle.

Consider the motion of the medium in the time interval $[0, T]$. Fix time intervals $0 = t_0 < t_1^n < \dots < t_n^n < T$, $\Delta t_i^n = t_{i+1}^n - t_i^n$, $i = 1, 2, \dots, n$. Introduce a system of functions u_0, u_1, \dots, u_n , where u_k minimize the functional $I(u)$ for a given u_{k-1} ,

$$I(u) = \int_{\Omega} \tau_0 |\nabla u| d\omega - \int_{\Gamma \cup \bigcup_1^N \Gamma_i} Q(t_k^n) u d\Gamma - \int_{\Omega} X(t_k) u d\omega +$$

$$+ \int_{\Omega} \rho \frac{(u - u_{k-1})^2}{2\Delta t_k^n} d\omega + \sum_{i=1}^N m_i \frac{(u - u_{k-1})^2}{2\Delta t_k^n} \Big|_{\Gamma_i} ; \quad u|_{\Gamma} = 0; \quad u|_{\Gamma_i} = \alpha_i. \quad (4)$$

Here α_i are constants; m_i is the mass of the cylinders immersed in the medium; ρ is the density of the medium.

Introduce the function $u_n(x, y, t)$, equal to $u_k(x, y)$ for $t = t_k^n$ and being the linear interpolation of the functions u_k and u_{k-1} on the interval $[t_{k-1}^n, t_k^n]$. If there exists a limit of the sequence of functions $u_n(x, y, t)$ as $n \rightarrow \infty$, independent of the manner of partitioning the interval $[0, T]$, then this limit is called the true motion of the ideally plastic medium.

It can be shown that the sequence $u_n(x, y, t)$ indeed converges to some function $u(x, y, t)$ on the interval $[0, T]$ as $\Delta t \rightarrow 0$, and $|u - u_n| < c \max_k \sqrt{t_k^n}$. The constant c can be found effectively.

It should be borne in mind, however, that passing to the limit as $\Delta t \rightarrow 0$ directly in (4) is impossible and the functional $I(u)$ has no corresponding analogue.

The variational principle formulated above is a generalization of Gauss' principle (3) to systems with an infinite number of degrees of freedom. For a system with a finite number of degrees of freedom it can be obtained directly from Gauss' principle by integration over an arbitrary small, but finite, time interval.

Consider the motion of an ideally plastic medium in a plane gap under the action of a time-dependent pressure gradient $C(t)$. Analysis of the solution of this problem makes it possible to illustrate clearly all the qualitative features characteristic of the motion of an ideally plastic medium, while avoiding cumbersome exposition.

Using (4) to solve the posed problem, after simple calculations we obtain that the ideally plastic medium will move in the gap as a rigid body with velocity

$$u(t) = \left\{ \frac{1}{\rho l} \int_0^t [lC(s) - 2\tau_0] ds \right\}^+, \quad (5)$$

where l is the width of the gap, and the plus sign at the brace denotes that only positive values of the integral are taken. If the integral in (5) becomes negative, then $u = 0$.

Formula (5) shows that if stationary motion exists, then $C(t) \rightarrow C^* = 2\tau_0/l$, and the velocity of the corresponding stationary motion is determined uniquely by the loading process. At the same time, stationary motion may also fail to exist if the integral in (5) diverges as $t \rightarrow \infty$. Note also that if, beginning from some instant of time, $C < C^*$, then the motion ceases in finite time. In the caserayele

case, if $C(t) = C^*$ for $t > 0$, while at $t = 0$ some impulse is imparted to the medium, the motion proceeds with a constant velocity depending on the magnitude of this impulse.

In conclusion, let us consider one more example of nonstationary motion of an ideally plastic medium. We return to the problem of the motion of two cylinders with square cross sections in an ideally plastic medium (Fig. 1). For $d = a$, taking the dynamics of the process into account completely removes the configurational nonuniqueness of the motion. Namely, the medium enclosed between the cylinders always has velocity $v_3 = 0$, while the velocities of the cylinders v_1 and v_2 are determined uniquely.

In the case when $d < a$ and $R_1 = R_2 = R(t)$,

$$\tau_0(3a + d) \leq R \leq \frac{2a\tau_0[(a - d)\rho_0 + 2\rho d]}{\rho d}, \quad 0 \leq t < T, \quad (6)$$

the cylinders and the medium enclosed between them move as a single whole with velocity

$$V(t) = \int_0^t \frac{R(s) - (3a + d)\tau_0}{\rho ad + 2\rho_0 a^2} ds. \quad (7)$$

If, however, for $T \leq t \leq T_1$

$$R(t) > 2a\tau_0[(a - d)\rho_0 + 2\rho d]/\rho d,$$

then the acceleration of the cylinders is greater and the velocities of motion of the cylinders and the medium are

$$v_1(t) = v(T) + \int_T^t \frac{R(s) - 4\tau_0 a}{\rho_0 a^2} ds; \quad v_2(t) = v(T) + \int_T^t \frac{2\tau_0(a - d)}{\rho ad} dt. \quad (8)$$

Here ρ_0 is the density of the material of the cylinders.

If, for $t > T_1$, the external force $R(t)$ again satisfies inequality (6), then formulas (8) remain valid until the velocities of the medium and of the cylinders become equal; thereafter the motion again proceeds according to (7).

Thus, the nonuniqueness of stationary motions of an ideally plastic medium is completely eliminated by taking into account the dynamics of the process by which the medium is brought into a regime of stationary flow.

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