ON THE PROBLEM OF STUDYING THE STABILITY OF FUNCTIONING OF COMPLEX SYSTEMS

V. V. KALASHNIKOV

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Abstract

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V. V. KALASHNIKOV

ON THE PROBLEM OF STUDYING THE STABILITY OF FUNCTIONING OF COMPLEX SYSTEMS

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Among the problems relating to the study of the general properties of individual classes of complex systems, an essential place is occupied by the study of the stability of their functioning. In what follows it will be assumed that the state of a complex system is described by a set of generalized coordinates z, and the process of its functioning is completely determined by the change of these coordinates in time, i.e., by functions $z(t,\alpha)$, $t \in T$, which are, generally speaking, realizations of some random process $Z(t,\alpha)$. Here T is the totality of the time instants under consideration, and α is an element of some abstract set Λ, which in what follows we shall call the set of parameters. A one-parameter family of functionals F_τ is given on the realizations $z(t,\alpha)$ for any $\alpha \in \Lambda$, i.e. $F_\tau = F_\tau\{z(t,\alpha), t \leqslant \tau; t, \tau \in T\}$. For fixed α and fixed realization $z(t,\alpha)$, the functional F_τ is a function of time τ, $\tau \in T$. In the space Ω of all possible real functions with domain T, select a class of sets $\mathcal{B}_\mu$, consisting of all sets B satisfying some property μ. Denote by $B(t)$ the range of values at the instant t of all functions belonging to the set B. Let $y(t)$ be some real function with domain T. We shall agree to distinguish types of inclusions $y(t) \in B$, determined by some parameter d. To each d there corresponds, generally speaking, some subset $T_d \subset T$. For definiteness, by the inclusion $y(t) \in B$ we shall understand that the function $y(t)$, $t \in T$, is an element of B. In this case there is no need to define the set T_0. By the inclusion $y(t) \in B$ we shall understand that the value of the function $y(t)$ is an element of $B(t)$; $T_1 = T$. In principle, other types of inclusions are possible as well, but for the purposes of the present note these two types suffice. In the set Λ we select a class of subsets $\mathcal{A}_\lambda$, consisting of all sets A satisfying the property λ. We shall agree to distinguish two numbers: a and $a-0$. In this case the inequality $b > a - 0$ is understood as $b \geqslant a$. The definition given below indicates the connection between the various previously introduced definitions of stability, although it is of methodological rather than practical interest.

Definition 1. A complex system is (λ,μ)-stable with respect to (ε_0, F, d),

$-0 \leqslant \varepsilon_0 \leqslant 1$, if for any $\varepsilon > \varepsilon_0$ and any set $B \in \mathcal{B}_\mu$ there exists a set $A \in \mathcal{A}_\lambda$ such that, for any $\alpha \in A$ and for all $\tau \in T_d$,

$$\mathfrak{P}\{F_\tau[z(t, \alpha),\, t \leqslant \tau;\, t \in T] \in B\} \geqslant 1 - \varepsilon.$$

The choice of different λ, μ, d, and ε_0 makes it possible to obtain different kinds of stability. Thus, if $\varepsilon_0 = 0$, then it is required that for any $\varepsilon > 0$ there exist a domain A in the parameter space from the class $\mathcal{A}_\lambda$ such that, for any $\alpha \in A$ and $\tau \in T_d$, the event $F_\tau \in B$ occur with probability $\geqslant 1 - \varepsilon$; if $\varepsilon_0 = -0$, then the existence of such a domain is required ⋯

for which, when $\tau \in T_d$, this event occurs with probability 1; if $\varepsilon_0 > 0$, then the definition requires that in the class $\mathcal{A}_\lambda$ there exist a set for which the event $F_\tau \in B$ is fulfilled with sufficiently high probability ($\geq 1 - \varepsilon_0$). The choice of particular d and ε_0 is determined by the nature of the problem under study. Let us illustrate this with several examples.

1. Let the system be described by the differential equations $dz/dt = \Phi(z, t)$. As the parameters of the system we shall take the initial deviations $z(0) \equiv \alpha$; $T = [0, \infty)$; the set Λ is the n-dimensional space, where n is the dimension of the vector z; the functional $F_\tau\{z(t, \alpha),\, t \leq \tau\} = \|z(\tau, \alpha)\|$, where $\|\cdot\|$ is some norm. The class $\mathcal{B}_\mu$ consists of all possible sets B_b, $b > 0$, of nonnegative functions defined on $[0, \infty)$ and bounded by the number b; $\mathcal{A}_\lambda$ is the class of sets containing some neighborhood of zero; $\varepsilon_0 = -0$, $d = 0$. In this case the definition of (λ, μ)-stability reduces to the definition of Lyapunov stability of the equilibrium at the origin of coordinates (1).
2. If, in the preceding example, the class $\mathcal{B}_\mu$ consists of a single set B of nonnegative functions bounded by a fixed number b, and $\mathcal{A}_\lambda$ is the class of sets containing a certain fixed set, then the definition of (λ, μ)-stability reduces to the definition of practical stability (1). In general, by choosing the properties λ and μ and the parameters α in an appropriate way, one can obtain any definition of stability given in (1).
3. Now let the system be described by the differential equations

$$dz/dt = \Phi[z, t, y(t, \beta)], \qquad \Phi(0, t, y) = 0,$$

where $y(t, \beta)$ is a parametric family of random processes, β is a parameter taking values from some set Λ_1 (2). As the parameter of the system we take the pair $(z(0), \beta)$, specified on the direct product of the sets $\Lambda_1 \times \Lambda_2 \equiv \Lambda$, where $z(0) \in \Lambda_2$. $\mathcal{A}_\lambda$ is the class of sets containing some neighborhood of the set $\{z(0) = 0\}$ in the space Λ; $\varepsilon_0 = 0$, $d = 1$. The class $\mathcal{B}_\mu$ and the functional F_τ are the same as in example 1. In this case the definition of (λ, μ)-stability reduces to the definition, given in (2), of stability in probability of the solution $z = 0$ of the given system of differential equations.

4. Consider queueing systems described in ⁽³⁾. Let the functional of the process of system operation be $F_\tau = W(\tau)$—the virtual waiting time until the start of service. The class $\mathcal{B}_\mu$ consists of one set B of uniformly bounded functions on T; $F_\tau \in B$, if there exists $B_b \subset B$ such that $F_\tau \in B_b$; $\varepsilon_0 = 0$. The class $\mathcal{A}_\lambda$ consists of the one-point fixed set $\{\alpha\}$. The set Λ may be of arbitrary nature. In this case the definition of (λ, μ)-stability reduces to the definition of substability given by Loynes.

As is known, sufficiently general models of complex systems are aggregative systems ⁽⁴⁾. Let us consider a class of aggregative systems—autonomous piecewise linear aggregates ⁽⁵⁾. The internal state $z(t)$ for a piecewise linear aggregate has the following structure: $z(t) \equiv (i, \eta_i)$, $i = 1, 2, ...$, are the principal states of the aggregate, and η_i is a vector of dimension m_i, whose components are called supplementary coordinates and vary in the domain Γ_i of an m_i-dimensional space according to the law $d\eta_i/dt = v_i$. When the vector η_i reaches the boundary γ_i of the domain Γ_i at the moment t, the internal state of the aggregate changes in accordance with the transition function

$$P_\alpha(G_j/\eta_i) = \mathcal{P}\{z(t+0) = (j, \eta_j),\ \eta_j \in G_j \subset \Gamma_j / z(t) = \\ = (i, \eta_i),\ \eta_i \in \gamma_i,\ \alpha \in \Lambda\}.$$

In [5] it was indicated that, for aggregates, one can single out a class of functionals of practical interest such that these functionals are additional coordinates and change according to laws analogous to the laws governing the change of the remaining coordinates. Below we study the stability of aggregates, which may be called practical stability by analogy with the concept introduced for differential equations [1], and give a method for determining the region of stability.

Definition 2. An aggregate *functions stably* with respect to $\rho \geqslant 0$ if, for given regions $B_i \subset \Gamma_i$, $i = 1, 2, \dots$, there exists a subset A in the set of parameters λ, possessing property λ, such that for any $a \in A$

$$\mathfrak{P}\{\eta(t) \in \bigvee_i B_i t < \infty\} \geqslant 1 - \rho.$$

Here $\eta(t)$ is understood as the vector $\eta_i(t)$, if at time t the aggregate is in the i-th principal state. It is obvious that Definition 2 is a particular case of Definition 1, with $\varepsilon_0 = \rho = 0$. Suppose that the aggregate parameters include the initial conditions $\eta(0)$, and by α we shall understand all the remaining parameters of the aggregate. Without loss of generality let us consider the case where, for some $i \in I$, $B_i = \Gamma_i$, while for the others B_i is the empty set. This can always be done if we assume that, when η_i reaches the boundary B_i lying inside Γ_i, the aggregate passes into a certain fictitious absorbing state ϕ. Consider the aggregate at the moments of its transition from one principal state to another. At these moments the states of the aggregate $z = (i, \eta_i)$ form a Markov process with a discrete parameter, with $\eta_i \in \gamma_i$. Let $\chi_\alpha^{(n)}(z)$ be the probability that within n transitions the aggregate will leave the set of admissible states I at least once, under the condition that $z(0) = z = (i, \eta_i),\ i \in I,\ \eta_i \in \gamma_i$. Then the functions $\chi_\alpha^{(n)}$ satisfy the recurrence relation

$$\chi_\alpha^{(1)}[(i,\eta_i)] = \sum_{j \notin I} P_\alpha(\Gamma_j/\eta_i),$$

$$\chi_\alpha^{(n+1)}[(i,\eta_i)] = \sum_{j \in I} \int_{\gamma_j} \chi_\alpha^{(n)}[(j,\eta_j)] P_\alpha(C(d\sigma_j)/\eta_i) + \chi_\alpha^{(1)}[(i,\eta_i)], \tag{1}$$

where $d\sigma_i$ is an elementary "area element" on the hypersurface γ_i, and

$$C(d\sigma_i) = \{\eta_i : \eta_i \in \Gamma_i, \eta_i = \eta_i^* + kv_i, \eta_i^* \in d\sigma_i,\ k \in (-\infty, 0]\}$$

is the part of the cylinder constructed on $d\sigma_i$ that lies in the region Γ_i.

The function

$$\chi_\alpha[(i,\eta_i)] = \lim_{n\to\infty} \chi_\alpha^{(n)}[(i,\eta_i)]$$

gives the probability that the aggregate will not leave the admissible region if $z(0) = (i, \eta_i)$. Then the region

$$A_1 = \{a,\ (i,\eta_i) : \chi_\alpha[(i,\eta_i)] < \rho\}$$

is the region of stability if it satisfies condition λ. Thus, the stability problem has been reduced to the estimation of a certain probability. If we assume that $\chi_\alpha[(i,\eta_i)] = 1$ for $i \notin I$, then the equation for χ_α can be written in the form

$$\chi_\alpha(z) = E_z \chi_\alpha, \tag{2}$$

where $E_z\chi_\alpha$ is the conditional mathematical expectation of the function χ_α. It is easy to show that χ_α is the minimal nonnegative solution of (2). Relations (1) can be written in a more convenient form if the aggregate is first reduced to the canonical form [5].

Solving equation (2) may prove difficult, although it allows one to find the entire region of stability. In this connection, the following method is proposed for estimating the function $\chi_\alpha(z)$, which is a certain analogue of Lyapunov' s method.

Theorem 1. Let there exist a function $V_\alpha(z) \geqslant 0$ such that $V_\alpha[(i,\eta_i)] \geqslant 1,\ i \notin I$; $E_{zV_\alpha} - V_\alpha(z) \leqslant 0,\ i \in I$ (it is assumed that E_{zV_α} exists). Then $V_\alpha(z) \geqslant \chi_\alpha(z)$ for all α, z.

Proof. $V_\alpha(z) \geq E_z V_\alpha \geq \chi_\alpha^{(1)}(z)$. Using (1), by induction we show that $V_\alpha(z) \geq \chi_\alpha^{(n)}(z)$. Consequently, $V_\alpha(z) \geq \chi_\alpha(z)$.

It is often of interest to estimate the probability $\chi_\alpha^{(n)}(z)$. For this purpose one may use the inequality

$$\chi_\alpha^{(n)}(z) \leq \chi_\alpha^{(k)}(z) + (n-k)\sup_z \left[\chi_\alpha^{(k+1)}(z) - \chi_\alpha^{(k)}(z)\right], \quad k < n, \tag{3}$$

which follows from relations (1). A more general theorem is also valid.

Theorem 2. *Let a Markov process with discrete parameter be given, taking at time n the value ζ_n from the space X and having transition probability at time n $p_\zeta^{(n)}(A)$, $A \subset X$. Define the random time $\tau(\zeta) = \inf\{n : \zeta(n) \notin Q, \zeta(0) = \zeta\}$ of the first exit from the set $Q \subset X$, if at the initial time the process was at the point ζ. Let $V(\zeta) \geq 0$ be a function on X; $V(\zeta) \geq 1$ for $\zeta \in X \setminus Q$; $E_\zeta^{(n+1)} V - V(\zeta) \leq \Delta_{n+1}$, $\Delta_{n+1} \geq 0$, $0 \leq n \leq N-1$, $\zeta \in Q$, where*

$$E_\zeta^{(n)} V = \int_X p_\zeta^{(n)}(d\xi) V(\xi).$$

Then

$$\chi^{(N)}(\zeta) \leq V(\zeta) + \mathcal{E}\left\{\sum_{i=1}^{\min(\tau,N)} \Delta_i \,/\, \zeta(0) = \zeta\right\} \leq V(\zeta) + \sum_{i=1}^{N} \Delta_i.$$

Proof. Denote by $N_1(\zeta) = \min[\tau(\zeta), N]$. Obviously,

$$\chi^{(N)}(\zeta) = \sum_{i=1}^{N} \int_Q p_\zeta^{(1)}(d\xi_1) \cdots \int_Q p_{\xi_{i-2}}^{(i-1)}(d\xi_{i-1}) \int_{X \setminus Q} p_{\xi_{i-1}}^{(i)}(d\xi_i) V(\xi_i). \tag{4}$$

Since

$$\int_{X \setminus Q} p_{\xi_{i-1}}^{(i)}(d\xi_i) V(\xi_i) = E_{\xi_{i-1}}^{(i)} V - \int_Q p_{\xi_{i-1}}^{(i)}(d\xi_i) V(\xi_i), \tag{5}$$

we have

$$\chi^{(N)}(\zeta) \leq V(\zeta) + \sum_{k=1}^{N} \Delta_k \left(\sum_{i \geq k} \mathcal{P}\{N_1 = i\}\right); \tag{6}$$

$$\sum_{k=1}^{N} \Delta_k \left(\sum_{i \geq k} \mathcal{P}\{N_1 = i\} \right) = \sum_{i=1}^{N} \mathcal{P}\{N_1 = i\} \left(\sum_{k=1}^{i} \Delta_k \right) = \mathcal{E} \left\{ \sum_{i=1}^{N_1} \Delta_i \,/\, \zeta(0) = \zeta \right\}. \tag{7}$$

It follows from (4)–(7) that

$$\chi^{(N)}(\zeta) \leq V(\zeta) + \mathcal{E} \left\{ \sum_{i=1}^{N_1} \Delta_i \,/\, \zeta(0) = \zeta \right\}.$$

Since $\sum_{i=1}^{N_1} \Delta_i \leq \sum_{i=1}^{N} \Delta_i$, we have

$$\mathcal{E} \left\{ \sum_{i=1}^{N_1} \Delta_i \,/\, \zeta(0) = \zeta \right\} \leq \sum_{i=1}^{N} \Delta_i.$$

The theorem is proved.

An analogous theorem, but under more stringent restrictions on the function V, was proved in [6]. The computed examples show that inequality (3) gives good estimates already for small k, and therefore it is advisable to take $\chi^{(k)}(z)$ for some k as the function $V(z)$.

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