A PRIORI ESTIMATES FOR THE RADII OF CURVATURE OF A CONVEX SURFACE

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.73

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

A PRIORI ESTIMATES FOR THE RADII OF CURVATURE OF A CONVEX SURFACE

In the present note we consider the question of a priori estimates for the principal radii of curvature of a closed convex surface F satisfying the condition

$$f(R_1, R_2) = \varphi(n), \qquad R_1 \geq R_2, \tag{1}$$

where R_1, R_2 are the radii of curvature, and n is the unit vector of the outer normal to the surface. Both functions f and φ are twice differentiable. The function f is assumed to be strictly monotone in both variables, i.e.

$$\partial f/\partial R_1 > 0, \qquad \partial f/\partial R_2 > 0. \tag{2}$$

Theorem 1. *At the point P_1 of the surface F where R_1 attains its maximum,*

$$(R_2 - R_1)\frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2}\frac{\varphi_s^2}{(\partial f/\partial R_2)^2} \geq \varphi_{ss}.$$

At the point P_2 of the surface where R_2 attains its minimum,

$$(R_1 - R_2)\frac{\partial f}{\partial R_1} + \frac{\partial^2 f}{\partial R_1^2}\frac{\varphi_s^2}{(\partial f/\partial R_1)^2} \leq \varphi_{ss}.$$

The differentiation of φ is performed along the arc s of the great circle in the corresponding principal direction.

We prove the first part of the theorem. Suppose that at the point P_1, $R_1 > R_2$. Then in a neighborhood of this point

$$f(R_1, R_2) \equiv g(R_1 R_2, R_1 + R_2),$$

where g is also a twice differentiable function. Thus equation (1), in a neighborhood of the point P_1, can be written in the form

$$g(R_1R_2, R_1+R_2)=\varphi(n). \tag{3}$$

Take the point P_1 of the surface as the origin of coordinates, and the principal directions on the surface at this point as the directions of the coordinate axes x, y. Let, for definiteness, the x-axis correspond to the principal direction with radius of curvature R_1. Denote by $H(x,y,z)$ the support function of the surface. Then, for $x=y=0,\ z=1$, we shall have

$$H=0, \qquad H_x=0, \qquad H_y=0.$$

Accordingly, for the function $h(x,y)=H(x,y,1)$,

$$h=0, \quad h_x=0, \quad h_y=0.$$

The sum and product of the principal radii of curvature of the surface F are expressed in the known manner through the second derivatives of the function H on the unit sphere[1]. If, in these expressions, we pass to the function $h(x,y)$, we obtain the values

$$\begin{gathered} R_1R_2=(rt-s^2)(1+x^2+y^2), \\ R_1+R_2=((1+x^2)r-2xys+(1+y^2)t)\sqrt{1+x^2+y^2}, \end{gathered} \tag{4}$$

where r, s, t are the generally accepted notations for the second derivatives of the function

h. Let us note that at the point P_1 itself, i.e., for $x=y=0$, $r=R_1$, $s=0$, $t=R_2$.

The Cartesian coordinate net on the plane $z=1$, under projection from the point P_1 onto the unit sphere $x^2+y^2+z^2=1$, becomes a curvilinear net. The coordinate lines $x=\text{const}$ and $y=\text{const}$ on the sphere are great circles. Draw the cylinder Z projecting the surface F onto the plane of the great circle $y=\text{const}$. The radius of curvature R of this cylinder along the line of contact with the surface is contained between the principal radii of curvature of the surface, i.e.,

$$R_2\le R\le R_1. \tag{5}$$

For the radius of curvature R we have the known expression

$$R=p+p_{ss},$$

where p is the value of the support function H on the unit circle $y=\text{const}$, and the differentiation is performed with respect to the arc of this circle. Denote the radius of curvature R, as a function of the coordinates x, y, by $w(x, y)$. Observing that

$$p = \frac{1}{\sqrt{1 + x^2 + y^2}} h(x, y),$$

we find for the function $w(x, y)$ the expression

$$w = r(1 + x^2 + y^2)^{3/2}/(1 + y^2).$$

Since the direction $y = 0$ at the point $(0, 0)$ corresponds to the principal direction R_1, we have $w(0, 0) = R_1$, and consequently, in view of inequality (5), the function $w(x, y)$ attains a maximum at the point $(0, 0)$. Therefore, at the point $(0, 0)$,

$$w_x = r_x = 0, \qquad w_y = r_y = 0,$$

$$w_{xx} = r_{xx} + 3r \leq 0, \qquad w_{yy} = r_{yy} + r \leq 0.$$

Differentiating equality (3) with respect to x at the point $(0, 0)$, we successively obtain

$$g_1(rt + rt_x) + g_2(r_x + t_x) = \varphi_x,$$

$$g_1(r_{xx}t + 2r_{xt}x + rt_{xx} + 4rt) + g_2(r_{xx} + t_{xx} + 3r + t) + g_{11}(r_{xt} + rt_x)^2$$
$$+2g_{12}(r_{xt} + rt_x)(r_x + t_x) + g_{22}(r_x + t_x)^2 = \varphi_{xx}.$$

From the first equality, noting that at the point $(0, 0)$, $r_x = 0$, $r = R_1$, $t = R_2$, we find

$$t_x = \frac{\varphi_x}{g_1 R_1 + g_2} = \varphi_x \Big/ \frac{\partial f}{\partial R_2}.$$

Accordingly the second equality is transformed into the form

$$w_{xx}\frac{\partial f}{\partial R_1} + w_{yy}\frac{\partial f}{\partial R_2} + (R_2 - R_1)\frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2}\frac{\varphi_x^2}{(\partial f/\partial R_2)^2} = \varphi_{xx}.$$

Hence, since $w_{xx} \leq 0$, $w_{yy} \leq 0$, we obtain the inequality

$$(R_2 - R_1)\frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2}\frac{\varphi_x^2}{(\partial f/\partial R_2)^2} \geq \varphi_{xx}.$$

It is not difficult to verify that

$$\partial x/\partial s = (1 + x^2 + y^2)/\sqrt{1 + y^2}.$$

Therefore at the point $(0, 0)$, $\varphi_x = \varphi_s$, $\varphi_{xx} = \varphi_{ss}$, and we obtain the inequality asserted by the theorem.

At the beginning of the proof we excluded the equality $R_1 = R_2$, requiring that at the point P_1, $R_1 > R_2$. But the inequality obtained is, obviously, also valid in

when $R_1 = R_2$. In this case it expresses the trivial fact that the function φ attains a maximum at the point P_1. The proof of the second part of the theorem on the minimum of R_2 is analogous, and therefore we omit it.

Let us consider two examples. Let $f(R_1, R_2) = R_1 R_2$. Then our inequality takes the form

$$(R_2 - R_1)R_1 \geq \varphi_{ss}.$$

Hence, observing that $R_1 R_2 = \varphi$, we obtain

$$R_1^2 \leq \varphi - \varphi_{ss}.$$

This estimate plays an important role in the solution of the well-known Minkowski problem on the existence of a closed convex surface with given Gaussian curvature as a function of the normal to the surface.

Let $f(R_1, R_2) = R_1 + R_2$. Then at the point of minimum of R_2 we shall have

$$(R_1 - R_2) \leq \varphi_{ss}.$$

Hence one obtains the lower estimate for R_2

$$2R_2 \geq \varphi - \varphi_{ss}.$$

This estimate is used in the solution of Christoffel' s problem on the existence of a closed convex surface with a prescribed sum of the principal radii of curvature [2].

From Theorem 1, as a corollary, one obtains the well-known theorem of A. D. Aleksandrov stating that if on a closed convex surface $f(R_1, R_2) = \text{const}$, then it is a sphere. Indeed, if this surface is not a sphere, then at the point P_1, $R_1 > R_2$. On the other hand, by Theorem 1, at this point $(R_2 - R_1)\partial f/\partial R_2 \geq 0$, i.e. $R_2 \geq R_1$, which is impossible.

From Theorem 1 one can obtain a general condition for the existence of a priori estimates for the radii of curvature of a closed convex surface satisfying condition (1). We shall say that for the functions f and φ condition $(*)$ is fulfilled if

$$\lim_{\substack{R_2 = R_2(R_1, n) \\ R_1 \to \infty}} \left\{ (R_2 - R_1)\frac{\partial f}{\partial R_2} + \frac{\partial^2 f}{\partial R_2^2} \frac{\varphi_s^2}{(\partial f / \partial R_2)^2} \right\} < \varphi_{ss}. \qquad (*)$$

We shall say that for these functions condition $(**)$ is fulfilled if

$$\lim_{\substack{R_1=R_1(R_2,n)\\ R_2\to\infty}} \left\{(R_1-R_2)\frac{\partial f}{\partial R_1}+\frac{\partial^2 f}{\partial R_1^2}\frac{\varphi_s^2}{(\partial f/\partial R_2)^2}\right\} < \varphi_{ss}. \qquad (**)$$

In these conditions $R_1(R_2,n)$ and $R_2(R_1,n)$ are the solutions of equation (1) with respect to R_1 and R_2, respectively.

Theorem 2. *If condition $(*)$ is fulfilled for the functions f and φ, then for the radii of normal curvature of the closed convex surface F satisfying equation (1) there exists an a priori estimate from above. If, however, condition $(**)$ is fulfilled for the functions f and φ, then for the radii of normal curvature there exists a positive estimate from below. These estimates depend only on the functions f and φ.*

In conclusion we note that the method presented for obtaining estimates is also applicable to the more general case when the function f also depends on n and equation (1) has the form

$$f(R_1,R_2,n)=\varphi(n).$$

Physico-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR

Received
16 I 1967

CITED LITERATURE

[1] W. Blaschke, *Differential Geometry and the Geometric Foundations of Einstein's Theory of Relativity*, Moscow–Leningrad, 1935, p. 222. [2] A. V. Pogorelov, *Uspekhi Mat. Nauk*, 8, no. 3 (55), 127 (1953).

Note: Figure translations are in progress. See original paper for figures.