ON SUFFICIENT CONDITIONS FOR AN ABSOLUTE MINIMUM

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Abstract

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ON SUFFICIENT CONDITIONS FOR AN ABSOLUTE MINIMUM

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Consider the problem of the absolute minimum of the functional

$$I(y(t), u(t)) = \int_{t_0}^{t_1} f^0(t, y, u)\, dt + F(y_0, y_1). \tag{1}$$

Here $y_0 = y(t_0)$, $y_1 = y(t_1)$; $y(t)$ is a vector-function, continuous and piecewise differentiable on the interval $[t_0, t_1] \subset T$ (T is the real line), with values in the n-dimensional real vector space Y. For all $t \in [t_0, t_1]$, the vector $t, y(t) \in B$, a given subset of the space $T \times Y$. The set of such functions $y(t)$ will be denoted by D_y. The vector-function $u(t)$ (r-dimensional) is defined on $[t_0, t_1]$ and is continuous everywhere on $[t_0, t_1]$, except for a finite number of points where it may have discontinuities of the first kind. For each $t, y(t) \in B$, the vector $u(t) \in Q(t, y)$, a given subset of the r-dimensional vector space U.

The conditions imposed on $y(t), u(t)$ define, for each fixed t, sets: $B(t)$, the admissible values of y, and $V(t)$, the admissible values of the pairs y, u.

In addition to the conditions listed above, the pair of functions $y(t), u(t)$ must satisfy the system of n differential equations

$$\dot{y} = f(t, y, u) \tag{2}$$

and the boundary conditions:

$$y_0 \in B_0 \subset B(t_0), \qquad y_1 \in B_1 \subset B(t_1).$$

The vector-function $f(t, y, u)$ (n-dimensional) and the function $f^0(t, y, u)$ are continuous in all arguments for $t \in [t_0, t_1]$, $y, u \in V(t)$; $F(y_0, y_1)$ is a continuous function defined on the set $B(t_0) \times B(t_1)$.

The totality of all pairs of vector-functions $y(t), u(t)$ having the properties listed above will be denoted by D.

The problem is formulated as follows. Find a sequence $\{\bar{y}_s(t), \bar{u}_s(t)\} \subset D$ such that the functional (1) on this sequence tends, as $s \to \infty$, to its least value (^{1}).

Analogous conditions were considered in the works (^{1},^{2}); however, in them stronger conditions are imposed on the function $\varphi(t, y)$, which can be weakened. This is made possible by Lemma 1. On the other hand, the refusal to consider the problem directly in the class D, as is done in (^{3},^{4}), makes it possible to broaden the range of problems under consideration and substantially weaken the a priori conditions introduced by the method of proof.

1. Basic lemma. Let sets M, N be given, and let a functional $I(v)$, $v \in M$, bounded below, be defined on the set M:

$$\inf_{v \in M} I(v) = m > -\infty.$$

It is possible to prove the following lemma, which is a generalization of the lemma of work (^{1}).

Lemma 1. In order that the sequence $\{\bar{v}_s\} \subset M$ minimize the functional I on the set M, it is sufficient that there exist a functional $L(v)$, $v \in N$, such that:

1) $I(\bar{v}_s) \to l_1 \leq l, \quad \bar{v}_s \in M, \quad l = \inf\limits_{v \in N} L(v);$
2) For any element $v \in M$ and any prescribed $\varepsilon > 0$ there exists an element $v^* \in N$ such that $L(v^*) - I(v) \leq \varepsilon$.

2. Sufficient conditions for optimality. Let the set B be open and let $y(t) \in D_y$; $y^*(t) = y(t) + \Delta y_0$, where $\Delta y_0 \in Y$ is a constant vector. Let, further, $P \subset B$ be a set of measure 0 in B, closed in B.

Lemma 2. For any neighborhood $W_0 \subset B(t_0)$ of the point $y(t_0)$ there exists a vector Δy_0 such that $y_0^* \in W_0$, the curve $t, y^*(t) \in B$, and it intersects the set P on a set of measure 0 on the interval $[t_0, t_1]$.

The proof is analogous to that given in (3) (pp. 491–493), but, in contrast to (3), the "shifted" curve is not a solution of equations (2).

Definition 1. We shall call a set $Q(x) \subset U$ continuous with respect to x, $x \in B$, if for any $u_1 \in Q(x)$, $u_2 \in Q(x + \Delta x)$ the following condition is fulfilled: for arbitrary $\varepsilon > 0$ there is a $\delta(x) > 0$ such that, if $\|\Delta x\| < \delta(x)$, $x + \Delta x \in B$, then there exist $u_1' \in Q(x + \Delta x)$, $u_2' \in Q(x)$ such that $\|u_1 - u_1'\| < \varepsilon$, $\|u_2 - u_2'\| < \varepsilon$.

Definition 2. We shall say that a set $Q(x) \subset U$, $x \in B$, satisfies the Lipschitz condition with respect to x, if for any $u_1 \in Q(x)$ and any Δx, $x + \Delta x \in B$, there exists $u_2 \in Q(x + \Delta x)$ such that $\|u_1 - u_2\| < \gamma \|\Delta x\|$, $0 \leq \gamma < \infty$. Here $\| \ \|$ is the Euclidean norm.

Definition 3. We shall say that a function $\varphi(t)$ satisfies the one-sided Lipschitz condition if

$$\varphi(t'') - \varphi(t') \leq K(t'' - t'), \qquad t'' \geq t'.$$

Let the set $Q(t, y)$ appearing in the formulation of the problem be nonempty for all $t, y \in B$ and continuous with respect to t, y in the sense of Definition 1. Introduce into consideration a scalar function $\varphi(t, y)$, continuous and continuously differentiable on $B \setminus P$, and locally satisfying the Lipschitz condition in B. Analogously to how this was done in (1), construct the functions

$$\Phi(y_0, y_1) = F(y_0, y_1) + \varphi(t_1, y_1) - \varphi(t_0, y_0),$$

$$R(t, y, u) = \varphi_y f(t, y, u) + \varphi_t - f^0(t, y, u).$$

Let the function $\varphi(t, y)$ be chosen so that the function

$$\mu(t) = \sup_{y,u} R(t, y, u), \qquad y \in B(t) \setminus P(t), \qquad u \in Q(t, y),$$

$$P(t) = \{y : \ t, y \in P\},$$

is summable and the number

$$\bar{\Phi} = \inf_{y_0, y_1} \Phi(y_0, y_1), \qquad y_0 \in B_0 \setminus P(t_0), \qquad y_1 \in B_1 \setminus P(t_1).$$

is finite.

Consider the set E^* of pairs of vector functions $y^*(t), u^*(t)$ such that the function $y^*(t)$ differs from the functions $y(t) \in D_y$ in that it may have discontinuities at the points t_0, t_1 and intersects the set P on a set of measure 0 on the interval $[t_0, t_1]$, while the function $u^*(t)$ is an arbitrary bounded function given on $[t_0, t_1]$, $u^*(t) \in Q(t, y)$ ($u^*(t)$ is not necessarily measurable).

On the set E^* define the functional

$$L(y^*(t), u^*(t)) = \Phi(y_0^*, y_1^*) - (\bar{P}) \int_{t_0}^{t_1} R(t, y^*(t), u^*(t))\, dt,$$

Here $(\overline{P}) \displaystyle\int_{t_0}^{t_1}$ is the upper Perron integral (5). The following holds.

Lemma 3. *For any pair $y, u \in D$ and any $\varepsilon > 0$ there exists a pair $y^*, u^* \in E^*$ such that $L(y^*, u^*) - I(y, u) \leqslant \varepsilon$.*

For the proof, one must take, as $y^*(t)$, curves of the form appearing in Lemma 2 and consider the indicated difference.

On the basis of Lemmas 1 and 3 the following is proved.

Theorem. *Let there be a sequence* $\{\overline{y}_s, \overline{u}_s\} \subset D$. *In order that it minimize the functional* I *on* D, *it is sufficient that there exist a function* $\varphi(t, y)$, *continuous, locally satisfying in* B *the Lipschitz condition, continuously differentiable everywhere on* $B \setminus P$, *such that*

$$I(\overline{y}_s, \overline{u}_s) \to \overline{\Phi} - \int_{t_0}^{t_1} \mu(t)\, dt, \qquad s \to \infty. \tag{3}$$

In the proof, as the set M one must take the set D, and as N the set E^*.

Remark 1. If the sequence $\{\overline{y}_s, \overline{u}_s\} \subset D \cap E^*$ is such that $|R(t, \overline{y}_s, \overline{u}_s)| \leqslant c(t)$, where $c(t)$ is some summable function, then condition (3) may be replaced by the conditions (^{1})

$$R(t, \overline{y}_s, \overline{u}_s) \to \mu(t) \quad \text{a.e.},$$

$$\Phi(\overline{y}_{0s}, \overline{y}_{1s}) \to \overline{\Phi}, \qquad s \to \infty.$$

Remark 2. If $f(t, y, u)$, $f^0(t, y, u)$ are differentiable on B with respect to t, y, and $Q(t, y) = Q(t)$, i.e. does not depend on y, then the conditions imposed on the function $\varphi(t, y)$ in the theorem can be weakened.

Namely, $\varphi(t, y)$ must be continuous and continuously differentiable on $B \setminus P$, satisfy a one-sided Lipschitz condition along any solution $y(t)$ of equation (2) (with piecewise-continuous $u(t)$) defined on the whole interval $[t_0, t_1]$ and passing through an arbitrary point of the set $B(t_0)$, and also be bounded on each compact subset of B. The function $\varphi(t, y)$ in this case may be discontinuous.

Instead of Lemma 2 one must use the proof of Lemma 4 from (^{3}); the proof of Lemma 3 is unchanged.

3. Estimate of the optimality of an arbitrary solution.

Let there be some function $\varphi(t, y)$ satisfying the requirements indicated above. Let also there be an arbitrary pair $\tilde{y}(t), \tilde{u}(t) \in D$ and $\Phi(\tilde{y}_0, \tilde{y}_1) = \overline{\Phi}$. This pair satisfies the estimate

$$\Delta(\tilde{y}(t), \tilde{u}(t)) = I(\tilde{y}, \tilde{u}) - m \leqslant \varphi(t_0, \tilde{y}_0) - \varphi(t_1, \tilde{y}_1) + \int_{t_0}^{t_1} \left[\mu(t) + f^0(t, \tilde{y}, \tilde{u})\right] dt.$$

The estimate is valid also in the case of a discontinuous $\varphi(t, y)$. Similar estimates for the case of a smooth $\varphi(t, y)$ were given by V. F. Krotov (^{6}).

4. The case of an infinite interval of integration.

Let $t_1 = \infty$. In this case the formulation of the problem requires certain refinements. The functions $y(t), u(t)$ must satisfy the conditions indicated above on any interval $[t_0, \tau] \subset [t_0, \infty)$.

Denote by A^ε the set of points $y^\varepsilon : \{|y^\varepsilon - y| < \varepsilon,\ y \in A \subset Y,\ y^\varepsilon \in Y\}$. Then the expression $y_1 \in B_1$ is understood in the following sense. For any $\varepsilon > 0$ there exists τ, $t_0 \leqslant \tau < \infty$, such that $y(t) \in B_1^\varepsilon$ for all $t > \tau$.

The set D is defined basically exactly as before, but an admissible pair $y(t), u(t)$ must satisfy the condition: $I(y, u)$ is defined and finite.

To obtain sufficient conditions for optimality it is necessary to impose additional requirements on the functions $f(t, y, u)$, $f^0(t, y, u)$, $\varphi(t, y)$ and on the set $Q(t, y)$. Namely, they must be continuous and satisfy a Lipschitz condition ($Q(t, y)$—in the sense of Definition 2) with respect to t, y, u for all $y(t) \in B^\varepsilon(t)$, $\varepsilon = \text{const}$, $u \in Q(t, y)$, $t \in [t_0, \tau]$, with a constant depending on t, y of the form $\Omega(\xi)$, $\xi = \|t, y\|^2$. The functions $f(t, y, u)$ must not grow "too fast," i.e. $\|1, f\| \leq \Omega(\xi)$.

Here the function $\Omega(\xi)$ must be continuous, continuously differentiable, and such that the function $\Omega'(\xi)/\Omega(\xi)$ is bounded, $0 \leq \xi < \infty$. As such a function one may take, for example, the functions $a + b\xi^q$, $ae^{b\xi}$, $q, a, b > 0$, q an integer. The function $\varphi_1 = \lim_{t\to\infty} \varphi(t, y)$ must be continuous for all $y_1 \in B_1^\varepsilon$. The set P must have measure 0 in B^ε, $\varepsilon = \text{const}$.

In this case the sufficient conditions are formulated in exactly the same way. In the proof of Lemma 2 it is necessary to take Δy not in the form $\Delta y = \Delta y_0 = \text{const}$, but, for example, in the form $\Delta y = \zeta e^{-t}\Delta y_0/\zeta + \Omega^3(\xi)$, $\zeta = \text{const} > 0$.

5. Some other generalizations. The sufficient conditions are completely generalized if the functions $f(t, y, u)$, $f^0(t, y, u)$ have a discontinuity on some set P_1 of measure 0 in B, closed in B, under the condition that all trajectories from D intersect P_1 in a finite number of points, or under the condition that any trajectory not satisfying this requirement can be approximated in D (in the sense of convergence with respect to I) by trajectories of this kind.

The optimality conditions are also generalized to the case when t_1 is not fixed, but the optimal value t_1 is assumed finite. In this case one must require

$$\overline{\Phi} = \inf_{t_1, y_0, y_1} \Phi(y_0, y_1, t_1), \qquad \mu(t) \equiv 0,$$

where t_1, $y_1 \in S \subset B$, S is given. This result for the case of smooth $\varphi(t, y)$ was communicated to me by V. F. Krotov.

In a similar way the same results are obtained, under the same conditions except for the Lipschitz condition on $\varphi(t, y)$ in the theorem, when instead of the set P one considers the piecewise-smooth set introduced in (2).

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Note: Figure translations are in progress. See original paper for figures.

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