ON THE FACTORIZATION OF MAPPINGS BY WEIGHT AND DIMENSION

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.831

MATHEMATICS

A. ARKHANGELSKII

ON THE FACTORIZATION OF MAPPINGS BY WEIGHT AND DIMENSION

(Presented by Academician P. S. Aleksandrov, 3 VIII 1966)

The purpose of this article is to describe a simple method of factorization of mappings. We shall first show it on the simplest example—by proving Mardešić' s factorization theorem. Then a simple general formalism will be indicated, and the general final results will become evident. In the last part of the work a number of applications are given to factorization theorems and the technique developed.

1. Theorem 1 (Mardešić). Let $f : X \to Y$ be a continuous mapping of a bicompactum X onto a bicompactum Y, and let $\dim X = n$ and the weight of Y not exceed τ. Then there exist a bicompactum Z and continuous mappings $h : X \to Z$ and $g : Z \to Y$ such that: 1) $f = gh$; 2) $\dim Z \leq n$; 3) the weight of $Z \leq \tau$.

Proof. Construction. Let B be a base of cardinality τ of the space Y, and let $\varphi = \{\gamma_\alpha : \alpha \in M\}$ be the family of all distinct finite covers of the bicompactum Y by elements of B. Put $\lambda_\alpha = \{f^{-1}U : U \in \gamma_\alpha\}$ and $\psi_0 = \{\lambda_\alpha : \alpha \in M\}$. For each pair of covers from ψ_0 take a finite open cover of the space X, star-refined into each of them, and of multiplicity not greater than $n+1$ on X. Denote the totality of the covers obtained by ψ_1. Applying to ψ_1 the operation described, define ψ_2, and so on—we obtain as a result a sequence $\{\psi_i\}$ of families of open finite covers of the bicompactum X, the cardinality of each of which is not greater than τ. Put

$$\psi^* = \bigcup_{i=1}^{\infty} \psi_i, \qquad \psi^* = \{\lambda_\beta : \beta \in M^*\}$$

and xRy, if

$$y \in A(x) = \bigcap_{\beta \in M^*} \lambda_\beta(x)^*.$$

It turns out that R is an equivalence relation; the quotient space $Z = X/R$ is the desired bicompactum (its elements are sets of the form

$$A(x) = \bigcap_{\beta \in M^*} \lambda_\beta(x));$$

the natural quotient mapping $X \to X/R$ is the desired h, and the mapping $g : Z \to Y$ is defined by the rule $g(A(x)) = f(x)$.

Verification. The mapping g is well-defined, for

$$A(x) = \bigcap_{\beta \in M^*} \lambda_\beta(x) \subseteq \bigcap_{\alpha \in M} \lambda_\alpha(x) = f^{-1}(x).$$

From the continuity of f and the perfection of h it follows that g is also continuous.

Obviously, $f = gh$. We show that $\dim Z \leq n$ and that the weight of $Z \leq \tau$. It is easy to see that

$$\mu_\beta = \{Z \setminus h(X \setminus V) : V \in \lambda_\beta\}$$

is an open cover of the space Z, and

$$\operatorname{ord} \mu_\beta \leq \operatorname{ord} \lambda_\beta \leq n + 1.$$

It is clear that

$$B^* = \bigcup_{\beta \in M^*} \mu_\beta$$

is a base of the topology of Z of cardinality not greater than τ. From the bicompactness of Z and the definition of the systems ψ^* and $\{\mu_\beta : \beta \in M^*\}$ it easily follows that into every open cover of the space Z there is inscribed some μ_β. This means that $\dim Z \leq n$. Mardešić’ s theorem is proved.

Remark. Briefly, the essence of the proof of Theorem 1 can be stated as follows: suppose a mapping $f : X \to Y$ and some refining sequence of covers of the space Y are given. We take the system of inverse images of these covers in X and supplement it to a system of covers of that of the same cardinality which, together with each pair of its elements, contains their star-refinement possessing the additional properties that interest us (multiplicity $\leq n+1$, cardinality finite). To this system there canonically corresponds the desired factorization. It is precisely this idea that will be exploited below.

2. It is convenient to give the following.

Definition. Let $\varphi = \{\gamma_\alpha : \alpha \in M\}$ and $\psi = \{\lambda_\beta : \beta \in L\}$ be two families of covers of a space X. By a refinement operation of the family φ with respect to ψ (briefly, an I-operation) we shall mean any mapping $I : M \times M \to L$ such that if $\beta = I(\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in M$, then λ_β is star-embedded both in γ_{α_1} and in γ_{α_2}. Systems of the form

$$I(\varphi|\psi) = \{\lambda_\beta : \beta \in I(M \times M)\}$$

---

[*] $\lambda_\beta(x)$ is the union of all elements of the cover λ_β that contain the point x.

[**] $\operatorname{ord} \lambda$ is the maximal number of elements of centered subsystems of λ.

play the principal role in what follows; we shall say that $I(\varphi|\psi)$ is a refinement of φ with respect to ψ. (Of course, both the refinement operation and the refinement itself are not uniquely determined by φ and ψ, and do not always exist.) What follows is based on the following elementary lemma.

3. **Lemma 1.** *Let X be a normal space; let A be its closed subset; let $\dim X \leq n$; and let γ be a locally finite open cover of the space X, with cardinality $\gamma \leq \tau$. Then there exists a locally finite open cover λ of the space X, star-embedded in γ, such that: 1) the cardinality $\lambda \leq \tau$, and 2) the multiplicity of λ at the points of the set A does not exceed $n+1$.*

Proof. Consider a mapping f of the space X onto a polyhedron P of weight τ such that the inverse image of some open cover η of the polyhedron P is inscribed in γ. Inscribe in η, in the star sense, some locally finite open cover η^* of cardinality $\leq \tau$. The cover $\gamma^* = f^{-1}\eta^*$ of the space X satisfies all requirements of the lemma except condition 2). Now inscribe in γ^* a locally finite cover μ of the set A by sets open in A, of multiplicity $\leq n+1$. For each $V \in \mu$, fix some set $\mathcal{U}(V)$, open in X, lying in an element of η^* and cutting out on A the set V (i.e. $V = \mathcal{U}(V) \cap A$). The system

$$\mu^* = \{\mathcal{U}(V) : V \in \mu\}$$

is inscribed in η^* and has the required multiplicity on A, but it may fail to be locally finite. In view of this, order the elements of the cover

$$\eta^* = \{G_\alpha,\ \alpha \in M\}$$

and put

$$\widetilde{\mathcal{U}}_\alpha = \bigcup\{\mathcal{U} : \mathcal{U} \in \mu^*,\ \mathcal{U} \subset G_\alpha \text{ and } \mathcal{U} \cap (X \setminus G_\beta) \neq \Lambda \text{ for every } \beta < \alpha\}.$$

It is clear that

$$\{\widetilde{\mathcal{U}}_\alpha : \alpha \in M\} \cup \{G_\alpha \setminus A : \alpha \in M\}$$

is the desired cover of the space X. The lemma is proved.

The following result is one of the two main ones. It is the final generalization of Theorem 1.

4. **Theorem 2 (Zarelua).** *Let $f : X \to Y$ be a continuous mapping of a bicompactum X onto a bicompactum Y, and suppose that the weight of Y does not exceed τ. Let, further, $\{F_\alpha : \alpha \in M\}$ be a family, of cardinality $\leq \tau$, of finite-dimensional closed subsets of the space X. Then there exist a bicompactum Z and continuous mappings $h : X \to Z$ and $g : Z \to Y$ such that: 1) the weight of $Z \leq \tau$; 2) $gh = f$; 3) $\dim hF_\alpha \leq \dim F_\alpha$.*

Proof. Construction. For each F_α, $\alpha \in M$, denote by ψ_α the family of all open finite covers of the space X of multiplicity at the points of the set F_α not greater than $(\dim F_\alpha + 1)$. Denote by χ some refining family, of cardinality τ, of finite open covers of Y, and put

$$\varphi = f^{-1}\chi = \{f^{-1}\gamma : \gamma \in \chi\},$$

where, of course,

$$f^{-1}\gamma = \{f^{-1}U : U \in \gamma\}.$$

Define by induction a sequence φ_i, putting

$$\varphi_0 = \varphi \quad \text{and} \quad \varphi_{k+1} = \bigcup_{\alpha\in M} I(\varphi_k|\psi_\alpha),$$

where $I(\varphi_k|\psi_\alpha)$ is an arbitrarily fixed refinement of φ_k with respect to ψ_α—the existence of such a refinement follows trivially from Lemma 3. Let

$$\varphi^* = \bigcup_{i=1}^{\infty} \varphi_i;$$

put xRy if

$$y \in \bigcap_{\lambda\in\varphi^*} \lambda(x).$$

Then R is an equivalence relation, and the factor space $X|R$ is the desired bicompactum. The mapping $h : X \to Z$ is the natural factor mapping, and $g : Z \to Y$ is defined by the formula

$$g(z) = f(\varphi^{-1}(z)).$$

The construction is complete, and the fact that the goal has been attained is verified in the same way as in the proof of Theorem 1. (For each $a \in M$ we take our confinal subfamily (with respect to star-refinement) of the family φ^*.)

Finally, the second main result. The final generalization of B. A. Pasynkov' s theorem from (3) is

5. **Theorem 3.** *Let X be a normal space; Y a metric space of weight $\leqslant \tau$; $f : X \to Y$ a continuous mapping, and $\{F_i : i = 1, 2, \ldots, \infty\}$ a countable family of closed finite-dimensional subspaces in X. Then there exist a metrizable space Z of weight $\leqslant \tau$ and continuous mappings $h : X \to Z$ and $g : Z \to Y$ such that $f = gh$, $\dim hF_i \leqslant \dim F_i$, and the weight of $Z \leqslant \tau$.*

Proof. Construction. Let φ be the inverse image of some countable refining sequence of open coverings of the space Y, the cardinality of each of which does not exceed τ. Denote by ψ_i the family of all open locally finite coverings of cardinality $\leqslant \tau$ of the space X, having on F_i multiplicity no greater than $(\dim F_i + 1)$. Put $\varphi_0 = \varphi$, and, assuming that the family of coverings φ_j for $j = 1, \ldots, k$ has already been defined, fix $I(\varphi_k \mid \psi_i)$ (we use here Lemma 1), $i = 1, 2, \ldots, \infty$, and put

$$\varphi_{k+1} = \bigcup_{i=1}^{\infty} I(\varphi_k \mid \psi_i) \quad \text{and} \quad \varphi^* = \bigcup_{k=1}^{\infty} \varphi_k.$$

Consider the relation xRy if and only if $y \in \bigcap_{\lambda\in\varphi^*} \lambda(x)$. Obviously, R is an equivalence relation; on the quotient set $Z = X/R$ we define a topology different from the quotient topology; a defining system of neighborhoods at a point $z \in Z$ is formed by the sets of the form $O^x_\lambda z$, where $x \in h^{-1}z$, $\lambda \in \varphi^*$, and $O^x_\lambda z$ is defined by the formula $O^x_\lambda z = \{z' : \lambda(x) \supset h^{-1}z'\}$. Z together with this topology is the desired space. The role of h is played by the natural mapping $X \to X/R$, and g is defined by the formula $g(z) = f(h^{-1}(z))$.

Verification. The system φ^* is countable; put $\mu(\lambda) = \{\mathrm{Int}(Z \setminus f(X \setminus V)) : V \in \lambda\}$ for each $\lambda \in \varphi^*$.

It is clear that $\psi^* = \{\mu(\lambda) : \lambda \in \varphi^*\}$ is a countable fundamental set of coverings of the space Z. Moreover, $\operatorname{ord}_z \mu(\lambda) \leqslant \operatorname{ord}_x \lambda$ for every $z \in Z$, $x \in h^{-1}(z)$, and every $\lambda \in \varphi^*$. Hence, for every $P_k = hF_k$ there is in ψ^* a confinal subfamily ψ^*_k, the elements of which, in some numbering, are successively star-refined into one another and have on P_k multiplicity $\leqslant (\dim F_k + 1)$. Applying Nagami' s theorem, we conclude that $\dim P_k \leqslant \dim F_k$. The rest is verified as before.

Some applications

6. **Corollary (Zarelua (4)).** *Let X be a normal space of weight $\leqslant \tau$, and let $\{F_\alpha : \alpha \in M\}$ be a family of its closed finite-dimensional subspaces of cardinality $\leqslant \tau$. Then there exists a bicompact extension bX of the space X such that: 1) the weight of $bX \leqslant \tau$; 2) $\dim [F_\alpha]_{bX} \leqslant \dim F_\alpha$ for every $\alpha \in M$.*

This is an obvious corollary of Theorem 2.

7. **Theorem 4.** *Let $f : X \to Y$ be an open mapping of a finite-dimensional bicompactum X onto a compactum Y. Then*

$$\dim\{y : y \in Y \text{ and } f^{-1}y \text{ is uncountable}\} \geqslant \dim Y - \dim X - 1^*.$$

Proof. By applying Theorem 1, find a compactum Z and continuous mappings $h : X \to Z$ and $g : Z \to Y$ for which: 1) $\dim Z = \dim X$; 2) $f = gh$. It is easy to see that the mapping g is then open. Theorem 7 for the case when X is a compactum was proved by me earlier (see (6)). Applying it to the mapping $g : Z \to Y$, we obtain the required conclusion.

8. **Theorem 5.** *Let $f : X \to Y$ be a continuous mapping of a weakly countably-dimensional bicompactum X onto an uncountable compactum Y. Then the set of all those points of Y which have an uncountable inverse image is uncountable.*

* If Y is not finite-dimensional, then we put $\dim Y = \infty$. Compare this theorem with the theorem of P. S. Aleksandrov from (1).

Proof. By the definition of weak countable-dimensionality, we have

$$X = \bigcup_{i=1}^{\infty} F_i,$$

where each F_i is finite-dimensional and closed in X. Applying Theorem 2, we reduce our problem to the case of a mapping of a weakly countable-dimensional compactum onto an uncountable compactum, which I considered earlier (see ([5])). In addition, using B. A. Pasynkov' s ingenious device and Theorems 2 and 3, we can obtain the known theorems on universal spaces (see ([3])) and some new ones, for example:

9. Theorem 6. *There exists a weakly countable-dimensional metric space of weight τ into which every weakly countable-dimensional metric space of weight $\leq \tau$ is homeomorphically embedded.*

10. Remark. Previously, the theorems of Mardešić and Pasynkov were proved by means of spectral techniques. With every directed-by-refinement family of covers one can associate a spectrum, and not a unique one. But this passage out of the space under consideration is connected with certain complications in notation, presentations, and arguments. Moreover, it makes no difference exactly which spectrum is associated with the directed-by-refinement system of covers (as this paper shows). Thus the delicate additional structure of a spectrum over a system of covers, consisting in the coordinated (for transitivity) fixation of certain inclusion relations, captures nothing new at the level of our interest. Zarelua' s proof also proceeded far by detour; in it the apparatus of rings was heavily used.

We conclude by formulating two unsolved problems:

11. Problem. *Is Theorem 4 true in the case when Y is an arbitrary bicompactum?*

12. Problem. *Is Theorem 5 true in the case when X is a countable bicompactum and Y is an uncountable bicompactum?*

Moscow State University
named after M. V. Lomonosov

Received
4 V 1966

CITED LITERATURE

[1] P. S. Aleksandrov, DAN, **4**, No. 7 (1936);
[2] S. Mardešić, Illinois J. Math., **4**, No. 2 (1960).
[3] B. A. Pasynkov, DAN, **154**, No. 5 (1964).
[4] A. V. Zarelua, DAN, **144**, No. 4 (1962).
[5] A. V. Arkhangel' skii, Matem. sborn., **69** (111), 1 (1966).
[6] A. V. Arkhangel' skii, Tr. Mosk. matem. obshch., **15**, 181 (1960).