Particular solution of an integro-differential equation with deviating argument of neutral type

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Abstract

A solution $x(t, \lambda)$ is called a singular solution of the operator equation $x(t) = \lambda P(x(t))$ if $x(t, \lambda) \to \infty$ for $\lambda \to 0$. Using the example of the equation

$$\begin{gathered}\dot{x}(t) = \lambda \int_0^1 [A_1(t,s)x(s-\tau) + A_2(t,s)\dot{x}(s-\tau) + A_3(t,s)x^2(s)]\,ds+ \\ \lambda^2 \int_0^1 [B_1(t,s)x(s) + B_2(t,s)x(s-\tau) + B_3(t,s)x(s-\tau)\dot{x}(s-\tau)]\,ds\end{gathered} \tag{1}$$

where $A_i(t,s)$, $B_i(t,s)$ are continuous functions in the square $0 \leq t,\ s \leq 1$; λ is a parameter; $0 < \tau < 1$ is a constant deviation, the question of the existence of a singular solution for a nonlinear integro-differential equation with a deviating argument of neutral type is investigated in the case where the integrand functions are polynomials with respect to the unknown function. Bibliography: 8 items.

Full Text

Preamble

In this section, we consider the asymptotic behavior of the solution $x(t, \lambda)$ to a class of nonlinear integral equations as the parameter $\lambda \to 0$. Building upon the foundational work in [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**], we investigate the case where the solution exhibits a singularity of the form $x(t) = \lambda^{-1}\psi_{-1}(t) + \psi_0(t) + ...$ Specifically, we analyze the equation:

$$x(t) = \lambda \int_0^1 [A_1(t,s)x(s-\tau) + A_2(t,s)x(s-\tau) + A_3(t,s)x^2(s)]ds$$
$$+ \lambda^2 \int_0^1 [B_1(t,s)x(s) + B_2(t,s)x(s-\tau) + B_3(t,s)x(s-\tau)x(s-\tau)]ds \tag{1}$$

where the initial condition is given by $x(t,\lambda) = \phi(t,\lambda)$ on the interval $E_0 = [-\tau, 0]$. We assume that the function $\phi(t,\lambda)$ can be expanded as:

$$\phi(t,\lambda) = \lambda^{-1}\phi_{-1}(t) + \phi_0(t) + \lambda\phi_1(t) + ... \tag{2}$$

We seek a solution in the form of a formal power series:

$$x(t,\lambda) = \sum_{k=-1}^{\infty} \lambda^k \psi_k(t) \tag{4}$$

Substituting expansion (4) into equation (1) and equating coefficients of like powers of λ, we obtain a system of recurrence relations for the functions $\psi_k(t)$. For the leading term $\psi_{-1}(t)$, we have:

$$\psi_{-1}(t) = \lambda_0 \int_0^1 A_3(t,s)\psi_{-1}^2(s)ds, \quad \psi_{-1}(t) = \phi_{-1}(t) \text{ on } E_0 \tag{5_{-1}}$$

For subsequent terms $\psi_k(t)$ where $k = 0, 1, 2, ...$, the equations take the form:

$$\psi_k(t) = \int_0^1 2A_3(t,s)\psi_{-1}(s)\psi_k(s)ds + F_k(\lambda_0, \psi_{-1}, \psi_0, \dots, \psi_{k-1}) \tag{5_k}$$

where F_k are known functions determined by the preceding terms of the expansion. Specifically, F_0 depends on the linear operators A_1 and A_2 acting on the delayed components of ψ_{-1}.

Solvability and Asymptotic Convergence

To ensure the existence of the expansion, we introduce the linear operator $L\psi = \psi(t) - \int_0^1 2A_3(t,s)\psi_{-1}(s)\psi(s)ds$. If λ_0 is not an eigenvalue of the kernel $2A_3(t,s)\psi_{-1}(s)$, the functions $\psi_k(t)$ are uniquely determined. We define the remainder of the series after K terms as $R_K(t,\lambda)$. Using the method of successive approximations and applying the contraction mapping principle in a suitable Banach space, we can establish the convergence of the series.

The error estimate for the K-th order approximation is governed by the functional $U(X, Y, \lambda)$, which satisfies:

$$|R_K(t, \lambda)| \leq C\lambda^{K+1}$$

where C is a constant independent of λ. This confirms that (4) is indeed an asymptotic expansion of the solution to the original problem (1)-(2).

Bifurcation and Branching of Solutions

In cases where the linear operator associated with (5_k) is singular (i.e., the Fredholm alternative condition is triggered), we encounter branching points. Let $w(t)$ be the eigenfunction corresponding to the kernel $2A_3(t, s)\psi_{-1}(s)$, and $v(t)$ be the eigenfunction of the adjoint operator. The solvability condition for $\psi_0(t)$ requires:

$$\int_0^1 \left[\phi_0(0) + \lambda_0 \int_0^1 F_0(z, s)ds\right] v(t)dt = 0 \tag{12}$$

If this condition is met, the general solution for $\psi_0(t)$ is given by $\psi_0(t) = C_0 w(t) + u_0(t)$, where C_0 is a constant to be determined from the solvability condition of the next equation in the hierarchy (7_1). This leads to a quadratic equation for C_0:

$$P_0 C_0^2 + Q_0 C_0 + T_0 = 0 \tag{13}$$

The roots of this equation determine the possible branches of the asymptotic solution. If $Q_1 \neq 0$, we obtain distinct branches C_{01} and C_{02}, leading to two different asymptotic expansions.

Conclusion

We have established the following results: 1. If λ_0 is not a characteristic value, the problem (1)-(2) possesses a unique asymptotic expansion of the form (4). 2. If λ_0 is a characteristic value and the solvability conditions are satisfied, the solution may branch. The coefficients C_k of these branches are determined uniquely by the subsequent equations in the hierarchy. 3. The formal series constructed are shown to be valid asymptotic representations of the actual solution $x(t, \lambda)$ as $\lambda \to 0$, with the error terms vanishing at the appropriate rates.

Figures

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DIFFERENTIAL EQUATIONS

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ON A SINGULAR SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION WITH A DEVIATING ARGUMENT OF NEUTRAL TYPE

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Let P be a certain operator. The solution $x(t, \lambda)$ of the equation $x(t) = P(x(t), \lambda)$ is called *singular* if $x(t, \lambda) \to \infty$ as $\lambda \to 0$.

Many mathematicians have been concerned with the question of the existence of singular solutions. For example, A. A. Temlyakov [1], H. N. Nazarov [2], M. M. Smirnov [3], P. P. Rybin [4], J. G. Yusif-zade [5] dealt with singular solus of integral equations, and the works of K. T. Akhmedova [6] and A. Iskenderov [7, 8] are devoted to singular solutions of integro-differential equations.

In our work, we investigate the question of the existence of a singular solution of a non-linear integro-differential equation with a deviating argument of neutral type in the case when the integrands are polynomials with respect to the unknown function. For simplicity of exposition, let us consider an equation of a particular form

$$\dot{x}(t) = \lambda \int_0^1 [A_1(t, s)x(s-\tau) + A_2(t, s)\dot{x}(s-\tau) + A_3(t, s)x^2(s)]ds +$$

$$+ \lambda^2 \int_0^1 [B_1(t, s)x(s) + B_2(t, s)x(s-\tau) + B_3(t, s)x(s-\tau)\dot{x}(s-\tau)]ds, \quad (1)$$

where $\dot{x}(s) = \frac{dx}{ds}$; $0 < \tau < 1$ — constant deviation; λ — parameter; $A_i(t, s)$, $B_i(t, s)$ — continuous functions of their arguments in the domain $D\{0 \leqslant t, s \leqslant 1\}$.

We will seek a solution $x(t, \lambda)$ of equation (1) for $0 \leqslant t \leqslant 1$ in the class C t of continuous functions having bounded derivatives, with the initial condition

$$x(t, \lambda) = \varphi(t, \lambda) \text{ on } E_0 = [-\tau, 0], \quad (2)$$

where $\varphi(t, \lambda)$ is continuous and has a bounded derivative.

Let us assume that the initial function $\varphi(t, \lambda)$ can be represented in the form of a series

$$\varphi(t, \lambda) = \frac{\lambda_0}{\lambda}\varphi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k \varphi_k(t), \quad (3)$$

where $\lambda_0 \neq 0$ is a yet unknown number, which will be determined later.

We will seek the solution of the problem (1), (2) in the form of a series

$$x(t, \lambda) = \frac{\lambda_0}{\lambda}\psi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k \psi_k(t). \quad (4)$$

Figure 1: Figure 1

Пodstabling series (4) into equattion (1) and cpabning koeffficieents at like powers of λ, we nolyarn the clellowing yquatienes for onpedemening kosф-fiqunentos series (4):

$$\dot{\psi}_{-1}(t) = \lambda_0 \int_0^1 A_3(t, s)\, \psi_{-1}^2(s)\, ds,$$

$$\psi_{-1}(t) = \varphi_{-1}(t) \text{ on } E_0, \qquad (5_{-1})$$

$$\dot{\psi}_k(t) = \lambda_0 \int_0^1 2A_3(t, s)\, \psi_{-1}(s)\, \psi_k(s)\, ds + F_k(\lambda_0, \psi_{-1}, \psi_0, \ldots, \psi_{k-1}), \quad (5_k)$$

$$\psi_k(t) = \varphi_k(t) \text{ on } E_0 \qquad (k = 0, 1, 2, \ldots),$$

rde

$$F_0 \equiv \lambda_0 \int_0^1 [A_1(t, s)\, \psi_{-1}(s - \tau) + A_2(t, s)\, \dot{\psi}_{-1}(s - \tau)]\, ds +$$

$$+ \lambda_0^2 \int_0^1 B_3(t, s)\, \psi_{-1}(s - \tau)\, \dot{\psi}_{-1}(s - \tau)\, ds,$$

$$F_1 \equiv \int_0^1 [A_1(t, s)\, \psi_0(s - \tau) + A_2(t, s)\, \dot{\psi}_0(s - \tau) + A_3(t, s)\, \psi_0^2(s)]\, ds +$$

$$+ \lambda_0 \int_0^1 \{B_1(t, s)\, \psi_{-1}(s) + B_2(t, s)\, \psi_{-1}(s - \tau) + B_3(t, s)[\psi_{-1}(s - \tau)\, \dot{\psi}_0(s - \tau) +$$

$$+ \dot{\psi}_{-1}(s - \tau)\, \psi_0(s - \tau)]\}\, ds,$$

$$F_k \equiv \int_0^1 \Big\{A_1(t, s)\, \psi_{k-1}(s - \tau) + A_2(t, s)\, \dot{\psi}_{k-1}(s - \tau) +$$

$$+ A_3(t, s) \sum_{i=0}^{k-1} \psi_i(s)\, \psi_{k-i-1}(s) +$$

$$+ B_1(t, s)\, \psi_{k-2}(s) + B_2(t, s)\, \psi_{k-2}(s - \tau) +$$

$$+ B_3(t, s)\Big[\lambda_0 (\psi_{-1}(s - \tau)\, \dot{\psi}_{k-1}(s - \tau) + \dot{\psi}_{-1}(s - \tau)\, \psi_{k-1}(s - \tau)) +$$

$$+ \sum_{i=0}^{k-2} \psi_i(s - \tau)\, \dot{\psi}_{k-i-2}(s - \tau))\Big]\Big\}\, ds$$

$$(k = 2, 3, \ldots).$$

Let us pass from problem (5_{-1}) to an awuivantent problem

$$\psi_{-1}(t) = \lambda_0 \int_0^t \int_0^1 A_3(u, s)\, \psi_{-1}^2(s)\, ds du + \varphi_{-1}(0), \qquad (6_{-1})$$

$$\psi_{-1}(t) = \varphi_{-1}(t) \text{ on } E_0.$$

Let λ_0 ad $A_3(t, s)$ be takuh, thıa problem (6_{-1}) umeet eduincessioue non-privit nonrial pelution $\psi_{-1}(t)$, ydobletropessing the yclorians organuvenions

$$|\psi_{-1}(t)| \leqslant a, \qquad |\dot{\psi}_{-1}(t)| \leqslant a.$$

Figure 2: Figure 2

For the determination of $\psi_k(t)$ $(k = 0, 1, 2, \ldots)$, we pass from prodlem (5_k) to an equivalent problem

$$\psi_k(t) = \lambda_0 \int_0^t \int_0^1 2A_3(u, s)\,\psi_{-1}(s)\,\psi_k(s)\,ds\,du + \int_0^t F_k(\lambda_0, \psi_{-1}, \psi_0, \psi_1, \ldots, \psi_{k-1})\,du + \varphi_k(0), \qquad (6_k)$$

$$\psi_k(t) = \varphi_k(t) \text{ on } E_0.$$

Memrning the onder of intergation in the douhble intergale (6_k), we obltain the ypabnome Fredholm

$$\psi_k(t) = \lambda_0 \int_0^1 \Phi(t, s)\,\psi_k(s)\,ds + \Psi_k(t), \qquad \psi_k(t) = \varphi_k(t) \text{ on } E_0, \qquad (7_k)$$

whe $\Psi_k(t) = \varphi_k(0) + \int_0^t F_k(\lambda_0, \psi_{-1}, \psi_0, \ldots, \psi_{k-1})\,du$ — is a continuous fynk- tion; $\Phi(t, s) = 2\psi_{-1}(s) \int_0^t A_3(u, s)\,du$ — is kernel, contvuous in D.

1. Lyct λ_0 not be a characteristic ѳne number of the intergraranoro ypabnerun

$$y(t) = \lambda_0 \int_0^1 \Phi(t, s)\,y(s)\,ds. \qquad (8)$$

Then, on the base of the well-mown theorem Fredholma, eaach of the sadiaw (7_k) umeet equictвennous non-nyree решenue, npunagлeжeting the classy C. Sequ- aennentally interripying ypabnerins (7_k), hañden

$$\psi_k(t) = \Psi_k(t) + \lambda_0 \int_0^1 R_{\lambda_0}(t, s)\,\Psi_k(s)\,ds, \qquad \psi_k(t) = \varphi_k(t) \text{ on } E_0,$$

где $R_{\lambda_0}(t, s)$ — is the resolvennt of krgna $\Phi(t, s)$. Thus, the koэфfficients pяda (4) определяются последовательпо и equinctвennын образоя.

Lyct R обозначate a чuclo, taken, чto

$$\left\{ \sup_D |R_{\lambda_0}(t, s)|;\ \sup_D \left| \frac{\partial R_{\lambda_0}(t, s)}{\partial t} \right| \right\} \leqslant R.$$

Trobly доkaate the exodgence pяda (4) and its nepwoй npousвodnoй no t достатoчпо доkaate the cxodgence pяda

$$\sum_{k=0}^{\infty} \lambda^k \psi_k(t), \qquad \sum_{k=0}^{\infty} \lambda^k \dot{\psi}_k(t). \qquad (9)$$

Для этого, рассмотpere cuctemy двух алгебрических уравнений

$$U_1(X, Y, \lambda) \equiv X - Q\left[\frac{1}{|\lambda_0|\,M a} \sum_{k=0}^{\infty} \lambda^k |\varphi_k(0)| + 2 + 2\lambda + |\lambda_0|\,a + \lambda(X + Y) + \frac{\lambda(X + Y + X^2) + \lambda^2(2X + XY)}{|\lambda_0|\,a} \right] = 0, \qquad (10)$$

Figure 3: Figure 3

$$U_8(X, Y, \lambda) = Y - Q\left[2 + 2\lambda + |\lambda_0| a + \right.$$

$$\left. + \lambda(X+Y) + \frac{\lambda(X + Y - X^2) + \lambda^2(2X + XY)}{|\lambda_0| a}\right] = 0, \qquad (10)$$

where

$$M \geqslant \{\sup_D |A_i(t, s)|;\ \sup_D |B_i(t, s)|\}, \qquad Q = |\lambda_0| Ma(1 + |\lambda_0| R).$$

Functions U_1 and U_2 at the point

$$X = a_0 = Q\left[\frac{|\varphi_0(0)|}{|\lambda_0| Ma} + 2 + |\lambda_0| a\right],$$

$$Y = b_0 = Q(2 + |\lambda_0| a), \qquad \lambda = 0$$

vanish, and the ctwo determinant $\dfrac{\partial(U_1, U_2)}{\partial(X, Y)}$ is non-zero at this point. Consequently, based on the implicit function theorem, the system (10) has a unique solution for X and Y, which is holomorphic with respect to the parameter λ, in the neighborhood of $\lambda = 0$:

$$X = \sum_{k=0}^{\infty} a_k \lambda^k, \qquad Y = \sum_{k=0}^{\infty} b_k \lambda^k. \qquad (11)$$

It is not difficult to prove that the series (11) majorize the series (9), respectively, and therefore, the series (9) converge absolutely and uniformly at least in the region where the series (11) converge.

Let us formulate the obtained result in the form of a theorem.

Theorem 1. If problem (6–1) has a non-trivial solution buhanging to class C, and λ_0 is not a characteristic number of the integral equation (8), then problem (1), (2) has a unique singular solution buihanging to class C and representable in the form of the series (4).

2. Let now λ_0 be a characteristic number of equation (8). For the sake of simplicity, we shall confine ourselves to the case where λ_0 is a characteristic number of the first rank.

Let us denote by $w(t)$ the eigenfunction of the kernel $\Phi(t, s)$, and by $v(t)$ the eigenfunction of the adjoint kernel, coorterstlying to the value λ_0. In this case, for the solvability of equation (7_0) it is necessary and sufficient that the condition

$$\int_0^1 \left\{\varphi_0(0) + \lambda_0 \int_0^t \int_0^t \left[A_1(z, s)\psi_{-1}(s-\tau) + A_2(z, s)\dot{\psi}_{-1}(s-\tau) + \right.\right.$$

$$\left.\left. + \lambda_0^2 B_3(z, s)\psi_{-1}(s-\tau)\dot{\psi}_{-1}(s-\tau)\right] dsdz\right\} v(t)\, dt = 0. \qquad (12)$$

Under condition (12), equation (7_0) has the solution

$$\varphi_0(t) = C_0 w(t) + u_0(t),$$

where C_0 is an arbitsary constant; $u_0(t)$ is a varticular solution of equation (7_0).

For the solvability of equation (7_1), it is necessary and sufficient that the condition

$$P_0 C_0^2 + Q_0 C_0 + T_0 = 0, \qquad (13)$$

be fullolned.

Figure 4: Figure 4

where

$$P_0 = \frac{1}{2\lambda_0} \int_0^1 \frac{v(s)\, w^2(s)}{\psi_{-1}(s)}\, ds;$$

$$Q_0 = \int_0^1 \int_0^t \int_0^1 \{A_1(z, s)\, w(s-\tau) + A_2(z, s)\, \dot{w}(s-\tau) +$$

$$+ \lambda_0 B_3(z, s)\, [\dot{w}(s-\tau) \cdot \psi_{-1}(s-\tau) + w(s-\tau) \cdot \dot{\psi}_{-1}(s-\tau)]\}\, v(t)\, dsdzdt +$$

$$+ \frac{1}{\lambda_0} \int_0^1 \frac{v(s)\, w(s)\, u_0(s)}{\psi_{-1}(s)}\, ds;$$

$$T_0 = \varphi_1(0) \int_0^1 v(t)\, dt + \int_0^1 \int_0^t \int_0^1 \{A_1(z, s)\, u_0(s-\tau) + A_2(z, s)\, \dot{u}_0(s-\tau) +$$

$$+ \lambda_0 [B_1(z, s)\, \psi_{-1}(s) + B_2(z, s)\, \psi_{-1}(s-\tau) + B_3(z, s)\, (\dot{u}_0(s-\tau)\, \psi_{-1}(s-\tau) +$$

$$+ u_0(s-\tau)\, \dot{\psi}_{-1}(s-\tau))]\}\, v(t)\, dsdzdt + \frac{1}{2\lambda_0} \int_0^1 \frac{v(s)\, u_0^2(s)}{\psi_{-1}(s)}\, ds.$$

Under the fulfillment of condition (13) the equation (7_1) will have a solution

$$\varphi_1(t) = C_1 w(t) + u_1(t), \qquad (14)$$

where C_1 is an arbitrary constant; $u_1(t)$ is a particular solution of equation (7_1).

Let's assume that equation (13) has two distinct roots C_{01}, C_{02} relative to C_0. Then

$$\varphi_{0i}(t) = C_{0i} w(t) + u_0(t) \qquad (i = 1,\ 2).$$

The constant C_1, entering into the solution (14), can be chosen from the solvabiility condition of equation (7_2):

$$Q_1 C_1 + T_1 = 0, \qquad (15)$$

where

$$Q_1 = \int_0^1 \int_0^t \int_0^1 \{A_1(z, s)\, w(s-\tau) + A_2(z, s)\, \dot{w}(s-\tau) +$$

$$+ \lambda_0 B_3(z, s)\, [\psi_{-1}(s-\tau)\, \dot{w}(s-\tau) + \dot{\psi}_{-1}(s-\tau)\, w(s-\tau)]\} \times$$

$$\times v(t)\, dsdzdt + \frac{1}{\lambda_0} \int_0^1 \frac{v(s)\, \psi_0(s)\, w(s)}{\psi_{-1}(s)}\, ds,$$

$$T_0 = \varphi_2(0) \int_0^1 v(t)\, dt + \int_0^1 \int_0^t \int_0^1 \{A_1(z, s)\, u_1(s-\tau) + A_2(z, s)\, \dot{u}_0(s-\tau) +$$

$$- B_1(z, s)\, \psi_0(s) + B_2(z, s)\, \psi_0(s-\tau) + B_3(z, s)\, [\lambda_0 (\psi_{-1}(s-\tau)\, \dot{u}_1(s-\tau) +$$

Figure 5: Figure 5

$$+ \dot{\psi}_{-1}(s-\tau)\,u_1(s-\tau)) + \psi_0(s-\tau)\,\dot{\psi}_0(s-\tau))\}\,v(t)\,dxdzdt +$$

$$+ \frac{1}{\lambda_0}\int_0^1 \frac{v(s)\,\psi_3(s)\,u_1(s)}{\psi_{-1}(s)}\,ds$$

— known numbers, sabrecurity ot C_{0i}.

Assuming $Q_1 \neq 0$, we find two values C_{1i} $(i = 1, 2)$ for C_1, C_1 with $C_{11} \neq C_{12}$ due to assumption $C_{01} \neq C_{02}$. Then

$$\psi_{1i}(t) = C_{1i}\omega(t) + u_1(t) \qquad (i = 1, 2).$$

If you substituting equation (15) obtaining (7_2) bydet inteet pelietion

$$\psi_2(t) = C_2\omega(t) + u_2(t),$$

where C_2 — a new arbitrary constant; $u_2(t)$ — private solution equation (7_2). Its solution polsabinnation equation (7_3)

$$Q_1C_2 + T_2 = 0$$

we find C_{2i} $(i = 1, 2)$. Continuing that process, the arbitrary constant C_n, extending is this solution $\psi_n(t) = C_n\omega(t) + u_n(t)$ equation (7_n), find-dem its solution polsabimation

$$Q_1C_n + T_n = 0$$

equation (7_{n+1}), where

$$T_n = \varphi_{n+1}(0)\int_0^1 v(t)\,dt + \int_0^1\int_0^t\int_0^1 \{A_1(z, s)\,u_n(s-\tau) + A_2(z, s)\,\dot{u}_n(s-\tau) +$$

$$+ B_1(z, s)\,\psi_{n-1}(s) + B_2(z, s)\,\psi_{n-1}(s-\tau) + B_3(z, s)\,\times$$

$$\times \left[\lambda_n\left(\psi_{-1}(s-\tau)\,\dot{u}_2(s-\tau) + \dot{\psi}_{-1}(s-\tau)\,u_n(s-\tau)\right) +\right.$$

$$\left.+ \sum_{i=0}^{n-1}\psi_i(s-\tau)\,\dot{\psi}_{n-1-i}(s-\tau)\right]\Big\}\,v(t)\,dsdzdt +$$

$$+ \frac{1}{2\lambda_0}\int_0^1 \frac{v(s)\left[2u_n(s)\,\psi_n(s) + \sum_{i=1}^{n-1}\psi_i(s)\,\psi_{n-1}(s)\right]}{\psi_{-1}(s)}\,ds.$$

Thus, the coefficients of the required series are determined sequentially and belong to class C.

Uniform convergence of the series

$$\sum_{k=0}^{\infty}\lambda^k\,\psi_{ki}(t) \qquad (i = 1, 2)$$

and their first derivatives with respect to t can be proved by the method of majorants.

Let us now consider the case when $Q_1 = 0$. In this case, the solution of the problem (1), (2) can be sought in the form of the series

$$x(t, \lambda) = \frac{\lambda_0}{\lambda}\,\psi_{-1}(t) + \frac{1}{\lambda^{\frac{2}{k2}}}\,\psi_0(t) + \sum_{k=1}^{\infty}\lambda^{\frac{b-1}{2}}\,\psi_k(t). \qquad (16)$$

Figure 6: Figure 6

For the determination of the coefficient $\psi_{-1}(t)$, as before, we obtain equation (6-1), for subsequent coefficients — the equations

$$\psi_0(t) = \lambda_0 \int_0^1 \Phi(t, s)\, \psi_0(s)\, ds, \qquad \psi_0(t) = 0 \text{ on } E_0, \tag{17_0}$$

$$\psi_k(t) = \lambda_0 \int_0^1 \Phi(t, s)\, \psi_k(s)\, ds +$$

$$+ \int_0^t F_k(\lambda_0, \psi_{-1}, \psi_0, \psi_1, \ldots, \psi_{k-1})\, dz + \varphi_{\frac{k-1}{2}}(0), \tag{17_k}$$

$$\psi_k(t) = \varphi_{\frac{k-1}{2}}(t) \text{ on } E_0.$$

wherein $\varphi_{\frac{k-1}{2}}(t)$ is considered equal to zero, if $\frac{k-1}{2}$ is a fractional number, where, in particular,

$$F_1 \equiv \int_0^1 \{\lambda_0 (A_1(z, s)\psi_{-1}(s-\tau) + A_2(z, s)\psi'_{-1}(s-\tau)) +$$

$$+ \lambda_0^2 B_3(z, s)\psi_{-1}(s-\tau)\psi'_{-1}(s-\tau)]\, ds,$$

$$F_2 \equiv \int_0^1 \{A_1(z, s)\psi_0(s-\tau) + A_2(z, s)\psi'_0(s-\tau) + 2A_3(z, s)\psi_0(s)\psi_1(s) +$$

$$+ \lambda_0^2 B_3(z, s)[\psi_{-1}(s-\tau)\psi'_0(s-\tau) + \psi'_{-1}(s-\tau)\psi_0(s-\tau)]\}\, ds,$$

$$F_3 \equiv \int_0^1 \{A_1(z, s)\psi_1(s-\tau) + A_2(z, s)\psi'_1(s-\tau) + 2A_3(z, s) \times$$

$$\times (\psi_0(s)\psi_2(s) + \psi_1^2(s)) + \lambda_0 (B_1(z, s)\psi_{-1}(s) + B_2(z, s)\psi'_{-1}(s-\tau)) +$$

$$+ B_3(z, s)[\lambda_0\psi_{-1}(s-\tau)\psi'_1(s-\tau) + \lambda_0\psi'_{-1}(s-\tau)\psi_1(s-\tau) +$$
$$+ \psi_0(s-\tau)\psi'_0(s-\tau)]\}\, ds,$$

$$F_4 \equiv \int_0^1 \{A_1(z, s)\psi_2(s-\tau) + A_2(z, s)\psi'_2(s-\tau) + 2A_3(z, s)(\psi_0(s)\psi_3(s) +$$

$$+ \psi_1(s)\psi_2(s)) + B_1(z, s)\psi_0(s) + B_2(z, s)\psi'_0(s-\tau) +$$

$$+ B_3(z, s)[\psi_0(s-\tau)\psi'_1(s-\tau) + \psi'_0(s-\tau)\psi_1(s-\tau) +$$

$$+ \lambda_0 (\psi_{-1}(s-\tau)\psi'_2(s-\tau) + \psi'_{-1}(s-\tau)\psi_2(s-\tau))]\}\, ds.$$

Let, as before, $w(t)$ be the eigenfunction of the kernel $\Phi(t, s)$, and $v(t)$ be the eigenfunction of the adjoint kernel, corresponding to the value λ_0.

Equation (17_0) has the solution

$$\psi_0(t) = C_0 w(t), \tag{18_0}$$

where C_0 is an arbitrary constant.

For the solvability of equation (17_1) it is necessary and sufficient for the condition to be fulfilled

Figure 7: Figure 7

$$\int \left\{\varphi_0(0) + \int\int \left[\lambda_0 (A_1(z,s)\psi_{-1}(s-\tau) + A_2(z,s)\dot\psi_{-1}(s-\tau) + \lambda_0^2 B_3(z,s)\psi_{-1}(s-\tau)\dot\psi_{-1}(s-\tau)\right] dsdz\right\} v(t)\, dt = 0. \tag{19_1}$$

Then equation (17_1) will have the solution $\psi_1(t) = C_1 w(t) + u_1(t)$, ($18_1$)

where C_1 is a new arbitrary constant, and $u_1(t)$ is a particular solution of the equation (17_1).

For the solvability of equation (17_2), it is necessary and sufficient to fulfill the condition $C_0 L_1 = 0$, (19_2)

where $L_1 = \int\int\int\int \{A_1(z, s)w(s-\tau) + A_2(z, s)\dot w(s-\tau) + 2A_3(z, s)(C_1 w(s) + u_1(t)) + \lambda_0 B_3(z, s)$
$[\psi_{-1}(s-\tau)\dot w(s-\tau) + \dot\psi_{-1}(s-\tau)w(s-\tau)]\}\, v(t)\, dsdzdt.$

Let us choose the constant C_1 such that $L_1 = 0$. Then the solution of the equation (17_2) will be the function (18_2), in which C_0 is an arbitrary constant, and the solution of the equation (17_3) will be the function $\psi_2(t) = C_2 w(t) + u_2(t)$, ($18_2$) where C_2 is a so far unknown constant, and $u_2(t)$ is a particular solution of the equation (17_2). Let us choose the constant C_2 from the condition of solvability of the equation (17_3): $L_2 C_2 + T_2 = 0$, (19_3)

where $L_2 = 2C_0 \int\int\int A_3(z, s)w(s)v(t)\, dsdzdt,$

$$T_2 = \int\int\int\int\int \{A_1(z, s)(C_1 w(s-\tau) + u_1(s-\tau)) + A_2(z, s) - A_2(z, s)(C_1 \dot w(s-\tau) + \dot u_1(s-\tau)) +$$
$$+ 2A_3(z, s)(C_1 w(s) + u_1(s))^2 + \lambda_0 (B_1(z, s)\psi_{-1}(s) + B_2(z, s)\dot\psi_{-1}(s-\tau)) +$$
$$+ B_3(z, s)[\lambda_0\psi_{-1}(s-\tau)(C_1\dot w(s-\tau) + \dot u_1(s-\tau)) + \lambda_0\dot\psi_{-1}(s-\tau)(C_1 w(s-\tau)$$
$$+ u_1(s-\tau)) + C_0 w(s-\tau)\dot w(s-\tau)]\}\, v(t)\, dsdzdt + \varphi_1(0) \int v(t)\, dt.$$

Assuming $L_2 \neq 0$, let us find C_2. Then equation (17_3) will have the solution $\psi_3(t) = C_3 w(t) + u_3(t)$. ($18_3$)

Figure 8: Figure 8

We determine the constant C_3 from the condition of solvability of equation (17_3):

$$L_2C_3 + T_3 = 0,$$

where T_3 is an expression depending on an arbitrary constant C_0 and previously found constants C_1, C_2.

Proceeding similarly, we determine the arbitrary constant C_n, e-entering into the solution

$$\psi_n(t) = C_n w(t) + u_n(t)$$

of equation (17_n), from the condition of solvability of equation (17_{n+1}):

$$L_2C_n + T_n = 0,$$

where T_n depends on an arbitrary constant C_0 and previously found values $C_1, C_2, \ldots, C_{n-1}$.

Thus, the coefficients of the series (16) are determined sequentially and are expressed through an arbitrary constant C_0. The convergence of series (16) and its derivative with respect to t can be proved by the method of majorants.

The obtained result is formulated in the form of the following theorem.

Theorem 2. *Let the problem* (6_{-1}) *have a non-trivial solution belonging to class C;* λ_0 *is a characteristic number of the first rank of equation* (8).

1) *If condition* (12) *is fulfilled, equation* (13) *has distinct roots,* $Q_1 \neq 0$, *then problem* (1), (2) *has two special solutions, representable in the form of series* (4).

2) *If equation* (13) *haes equal roots or one root (in this case the function* $A_3(t, s)$ *is such that* $v(t)$ *and* $\dfrac{w^2(t)}{\psi_{-1}(t)}$ *are orthogonal on* [0, 1], *i.e.* $P_0 = 0$), *then for* $Q_1 \neq 0$ *problem* (1), (2) *has a unique solution, representable in the form of series* (4).

3) *If if* $Q_1 = 0$, *condition* (19_1) *is fulfilled,* $L_2 \neq 0$, *then problem* (1), (2) *has a one-parameter family of special solutions, representable in the form of series* (16).

3) If Q $Q_1 = 0$, condition (19_1) is fulfilled, $L_2 \neq 0$, then problem (1), (2) has a *one-parameter family of* special solutions, representable in the form of series (16).

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Figure 9: Figure 9