ON SQUARE ROOTS OF METRIC AUTOMORPHISMS

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ON SQUARE ROOTS OF METRIC AUTOMORPHISMS

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1. The problem of extracting a square root from a metric automorphism has been solved for automorphisms with discrete spectrum ([1]), quasidiscrete spectrum ([2]), and shift automorphisms of the space of realizations of stationary Gaussian processes. The only obstacle to extracting a square root in the case of discrete spectrum is the eigenvalue -1. It is not difficult to verify that automorphisms with quasidiscrete spectrum may fail to have square roots even if -1 is not among the eigenvalues. The natural question arises: can a square root be extracted from every automorphism with continuous spectrum?

The present note contains: 1) the construction of an automorphism with continuous spectrum that has no square root*; 2) some theorems on square roots of automorphisms with simple spectrum.

The work makes essential use of the method of approximating automorphisms by periodic transformations ([3]). Let us recall the corresponding definitions.

Let (X, μ) be a Lebesgue space ([5]). We shall say that a sequence of measurable partitions ξ_n ([5]) of the space X tends to ε, and write $\xi_n \to \varepsilon$ (ε is the partition of the space X into points), if for every measurable set $M \subset X$ and every $\delta > 0$ there is a number N_δ such that for every $n > N_\delta$ there exists a set M_n, measurable with respect to the partition ξ_n, for which $\mu(M \Delta M_n) < \delta$.

Let T be an automorphism of the space (X, μ), and let $\varphi(q)$ be a sequence of positive numbers, where q is a natural number.

Definition 1. We shall say that the automorphism T **admits cyclic approximations with rate** $\varphi(q)$, if there exists a sequence $\{\xi_n, T_n\}$ of finite measurable partitions ξ_n of the space (X, μ) and cyclic automorphisms T_n such that

A.1. $\xi_n \to \varepsilon$.

A.2. $T_n \xi_n = \xi_n$.

A.3.
$$\sum_{i=1}^{q_n} \mu(T C_n^i \Delta T_n C_n^i) < \varphi(q_n).$$

Here C_n^i are the elements of the partition ξ_n, $i = 1, 2, \dots, q_n$, and q_n is the number of elements of ξ_n.

We shall use the following assertion.

Theorem 1 (V. I. Oseledets). *If an automorphism T admits cyclic approximations with rate $o(1/q^2)$, then T has simple spectrum.*

2. Z_2-extensions.

Definition 2. We shall say that an automorphism T of a Lebesgue space (X, μ) **admits a Z_2-extension** if there exists an involutive automorphism I of the space (X, μ), commuting with T and having no fixed points mod 0. The automorphism I will be called a **Z_2-extension of the automorphism** T.

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* *Note added in proof.* After this note had been submitted for publication, an article by Chacon [7] appeared, in which a different construction of an automorphism with continuous spectrum and without a square root was proposed.

Let ξ be a partition of (X, μ) whose elements are pairs $\{x, Ix\}$, $x \in X$. Put $\pi : X \to X/\xi = Y$, $T/\xi = S$. Let $X = X_1 \cup X_{-1}$, $IX_1 = X_{-1}$. Denote by C the mapping $x \to (\pi x, i)$, if $x \in X_i$, $i = \pm 1$. C is an isomorphism of the Lebesgue space X and $Y \times Z_2$, where Z_2 is the two-point set $\{\pm 1\}$ with measures $(1/2, 1/2)$. The automorphism CTC^{-1} has the form $(y, i) \to (Sy, n(y)i)$, where $n(y)$ is a function on Y with values ± 1. Such an automorphism will be called a **skew product** with base S and function n (see (4)).

Let ξ_n be a sequence of finite measurable partitions of the space (X, μ) tending to the partition ε into individual points, q_n the number of elements of ξ_n, and $\varphi(q)$ a sequence of positive numbers.

Definition 3. A measurable set $M \subset X$ is **oddly approximated** with respect to $\{\xi_n\}$ at rate $\varphi(q)$ if there exist a sequence of natural numbers $\{n_k\}$, $n_k \to \infty$, and a sequence of sets $\{M_k\}$ such that M_k consists of an odd number of elements of ξ_{n_k} and

$$\mu(M \triangle M_k) < \varphi(q_{n_k}).$$

Let T be a skew product with base S and function n.

Lemma 1 (see [3]). *If S admits cyclic approximations $\{\xi_n, S_n\}$ at rate $o(1/q)$ and $n^{-1}(-1)$ is oddly approximated with respect to $\{\xi_n\}$ at the same rate, then T is ergodic.*

Lemma 2. *If S admits cyclic approximations at rate $o(1/q^2)$ and $n^{-1}(-1)$ is a set oddly approximated with respect to $\{\xi_n\}$ at the same rate, then T has simple spectrum.*

The proof follows from Theorem 1.

In what follows we shall use the fact of the existence of an automorphism with continuous spectrum that admits cyclic approximations of odd orders at rate $o(1/q^2)$. We briefly describe the construction of such an automorphism. Let the number α, $0 < \alpha < 1$, be such that

$$|p_k/q_k - \alpha| = o(1/q_k^3),$$

where p_k/q_k are the convergents of α, $q_{2n} = 4q'_{2n}$, q'_{2n} is odd, and q_{2n+1} is not divisible by 4. Let T be the derived transformation ([6]) on an arc of length 3/4, induced by a rotation of the circle of unit length through angle α. The proof that T satisfies the stated requirements can be extracted from ([3]).

We shall denote by U_T the unitary operator in $L^2(X,\mu)$ associated with the automorphism T of the space (X,μ).

Theorem 2. *An ergodic automorphism T of a Lebesgue space (X,μ) admits a Z_2-extension if and only if in the unitary ring $L^2(X,\mu)$ there exists a multiplicative involutory unitary operator $V \neq E$ commuting with U_T.*

Theorem 3. *If an automorphism T with simple spectrum admits a Z_2-extension I, then $\sqrt{T}$ (if such exists) also admits the extension I.*

Proof. Both U_I and $U_{\sqrt{T}}$ commute with U_T. Since U_T has simple spectrum, U_I and $U_{\sqrt{T}}$ are functions of U_T, and consequently U_I commutes with $U_{\sqrt{T}}$. By Theorem 2 we obtain that $\sqrt{T}$ admits the Z_2-extension I.

We give a useful reformulation of Theorem 2.

If T has simple spectrum and is a skew product with base S, then a square root of T should be sought in the class of skew products with base $\sqrt{S}$, where $\sqrt{S}$ ranges over all possible square roots of S.

3. Construction of an automorphism with continuous spectrum having no square root. Let R be an automorphism of (X,μ) with continuous spectrum, admitting cyclic approximations $\{\xi_n, R_n\}$ of odd orders at rate $o(1/q^2)$. Let S be a skew product with base R and function n. Require that the sets $n^{-1}(1)$ and $n^{-1}(-1)$ be oddly approximated with respect to ξ_n at rate $o(1/q^2)$. We shall show that S has simple continuous spectrum. By Lemma 1, S is ergodic. If $Y = X \times Z_2$, $y = (x,i)$, $i = \pm 1$,

$f(y) = f(x,i) = f_1(x) + i f_{-1}(x)$ and $U_S f = \xi f$ with $\xi \neq 1$, then $n(x) f_{-1}(Rx) = \xi f_{-1}(x)$. Squaring, we obtain $f_{-1}^2(Rx) = \xi^2 f_{-1}^2(x)$. But R has continuous spectrum, and therefore $\xi^2 = 1$; in the case $\xi = -1$ we have $-n(x) f_{-1}(Rx) = f_{-1}(x)$, which contradicts the ergodicity of the skew product with base S and function $-n(x)$. The simplicity of the spectrum of S follows from Lemma 2.

Consider the skew product T with base S and function $m(y) = m(x,i) = i$. Clearly, $m^{-1}(1)$ and $m^{-1}(-1)$ are unevenly approximated with respect to the sequence of partitions $\eta_n = \xi_n \times Z_2$ of the space $Y = X \times Z_2$. Similarly to the preceding, we obtain that the automorphism T has simple continuous spectrum.

By Theorem 2, the square root of T must be sought in the form of a skew product with base $\sqrt{S}$ and some function α satisfying the equation

$$\alpha(\sqrt{S}y)\alpha(y) = m(y) = m(x, i) = i. \tag{1}$$

Let $I(x, i) = (x, -i)$, $i = \pm 1$. Then $m(Iy) = -m(y)$ and

$$\alpha(\sqrt{S}Iy)\alpha(Iy) = -m(y). \tag{2}$$

Multiplying (1) and (2), we have

$$\alpha(y)\alpha(Iy)\alpha(\sqrt{S}y)\alpha(\sqrt{S}Iy) = -1. \tag{3}$$

Denote $\alpha(y)\alpha(Iy) = \beta(y)$. Since S has simple spectrum, I and $\sqrt{S}$ commute and, consequently, $\alpha(\sqrt{S}y)\alpha(\sqrt{S}Iy) = \beta(\sqrt{S}y)$. Then (3) takes the form $\beta(y)\beta(\sqrt{S}y) = -1$. Hence it follows that $\beta \neq \text{const}$. Moreover, $\beta(Sy) = -\beta(\sqrt{S}y) = \beta(y)$, which contradicts the ergodicity of S. Thus, T has no square root.

In the construction given above we needed a function n such that $n^{-1}(1)$ and $n^{-1}(-1)$ are unevenly approximated with respect to some sequence of partitions ξ_n. We formulate a proposition showing that there are "sufficiently many" such functions. Let (X, μ) be a Lebesgue space, and let n and m be measurable functions on X taking the values ± 1. Put $\rho(n, m) = \mu(n^{-1}(1)\Delta m^{-1}(1))$. Denote by B the metric space of functions on X with values ± 1 and metric ρ.

Lemma 3. Let ξ_n be a sequence of partitions of the space (X, μ) tending to ε, and let $\varphi(q)$ be a sequence of positive numbers. The set of functions n such that $n^{-1}(1)$ and $n^{-1}(-1)$ are unevenly approximated with respect to ξ_n with rate $\varphi(q)$ is an everywhere dense set of second category in B.

With the aid of Lemma 3 one proves

Theorem 4. Let S be an automorphism admitting cyclic approximations $\{\xi_n, S_n\}$ of odd orders with rate $o(1/q^2)$, and let S^2 admit no Z_2-extension. Then in the space of skew products with fiber Z_2 and base S^2, the set of automorphisms having no square root is an everywhere dense set of second category in the natural topology induced by the metric ρ.

We omit the proof.

4. Let an automorphism T of the space (Y, ν) be a skew product with base S and function n, where S is an automorphism of the space (X, μ), $Y = X \times Z_2$, $y = (x, i)$, $x \in X$, $i = \pm 1$. Then $L^2(Y, \nu) = H_1 \oplus H_{-1}$, where H_1 is the subspace of functions such that $f(x, i) = f(x, -i)$; H_{-1} is the subspace of functions satisfying the condition $f(x, -i) = -f(x, i)$. Let σ_i be the maximal spectral type of T in H_i, $i = \pm 1$; let I be the natural Z_2-extension of the automorphism T.

Theorem 5. If $\sigma_1 \perp \sigma_{-1}$, then $\sqrt{T}$ (if it exists) admits the Z_2-extension I.

Ergodic skew products with a layer Z_2 over automorphisms with discrete spectrum satisfy the condition of Theorem 5.

If the automorphism S is ergodic, then the following is true.

Theorem 6. *If $\sigma_1 \perp \sigma_{-1}$, then T is the ergodic square of some automorphism if and only if, for some root $\sqrt{S}$, there is a representation*

$$n(x) = a(\sqrt{S}x)a(x),$$

and the skew product with base $\sqrt{S}$ and function a is ergodic.

5. Let us establish the connection between square roots and Z_2-extensions of an automorphism with simple spectrum.

Lemma 4. *If S_1 and S_2 are two distinct roots of an automorphism T with simple spectrum, then T admits a Z_2-extension; b) if T has simple spectrum and admits a Z_2-extension I, $S^2 = T$, then IS is a square root of T.*

Remark 1. *Part b) of Lemma 4 can also be proved under the more general assumption of Theorem 5.*

Thus, either an automorphism T with simple spectrum has no root, or it has a unique root and admits no Z_2-extension, or T has both a root and a Z_2-extension.

Theorem 7. *An automorphism T with simple spectrum either has no square roots, or their number is equal to a power of two, or there are infinitely many of them.*

Proof. We shall regard the identity automorphism E as the trivial Z_2-extension. Then the set of Z_2-extensions of the automorphism T forms an abelian group G. If G is infinite and T has a root, then, by Lemma 4, T has infinitely many roots. If G is finite, then G decomposes into a direct sum of cyclic groups of orders $\tau_1, \dots, \tau_n$, where τ_{i+1} is divisible by τ_i. But all elements of the group G have prime order 2; therefore

$$G = Z_2 + Z_2 + \dots + Z_2$$

(n summands). Consequently, the order of G is 2^n. Thus the number of nontrivial Z_2-extensions of the automorphism T is $2^n - 1$, and, if T has a root, then by Lemma 5 there are 2^n roots in all.

Remark 2. *The proof of Theorem 7 generalizes to the case of roots of prime degree, with the number 2 replaced by the corresponding prime number in the statement of the theorem.*

Remark 3. In Theorem 7 we found the number of "geometrically" distinct roots of an automorphism T with simple spectrum. It turns out that all of them are spectrally nonisomorphic. Indeed, let

$$VU_{\sqrt{T}}V^{-1} = U_{IU_{\sqrt{T}}};$$

I is a Z_2-extension of the automorphism T; V is a unitary operator. Then

$$VU_{\sqrt{T}}V^{-1}VU_{\sqrt{T}}V^{-1} = U_{IU_{\sqrt{T}}}U_{IU_{\sqrt{T}}}, \qquad VU_T = U_{TV},$$

i.e. V is a function of U_T, and consequently

$$VU_{\sqrt{T}} = U_{\sqrt{T}}V, \qquad U_{\sqrt{T}} = VU_{\sqrt{T}}V^{-1} = U_{IU_{\sqrt{T}}},$$

$U_I = E$, which contradicts the nontriviality of the extension I.

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Note: Figure translations are in progress. See original paper for figures.