ON DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS AND SYSTEMS

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 517.946.9

MATHEMATICS

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ON DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS AND SYSTEMS

(Presented by Academician I. G. Petrovskii on 19 III 1966)

Recently many works have appeared devoted to boundary-value problems for nonlinear elliptic equations and systems of higher orders. In the works of M. I. Vishik ([1]), F. Browder ([2]), and others, boundary-value problems were considered for a broad class of quasilinear elliptic equations and systems of arbitrary order. In the present note we consider a certain class of degenerate elliptic equations and systems. For the first boundary-value problem an iterative process is constructed which, in the case of existence, leads to the unique solution of the problem. In order to obtain the iterations, at each step one must solve a linear problem corresponding to the first boundary-value problem for the Poisson equation in the case of a second-order equation. This method for quasilinear problems with bounded nonlinearities was considered by us in ([3])*. For degenerate second-order equations the convergence of the method was established in our work ([4]), under the condition that there exists a bounded $C^{(1)}$ solution of the problem under consideration. In the present note we consider more general systems under weaker restrictions on the coefficients and the solution.

Let Ω be a domain of m-dimensional Euclidean space, bounded by a sufficiently smooth closed surface Γ. Consider inside Ω a system of equations with respect to the vector function $u = (u_1, \dots, u_N)$

$$L(u) \equiv \sum_{0\le|\alpha|,|\beta|\le r} (-1)^{|\alpha|} D^\alpha \left[a_n(x; u, \dots, D^\beta u)\right] = 0, \tag{1}$$

where α, β are multiindices

$$\alpha = (\alpha_1, \dots, \alpha_m), \quad \beta = (\beta_1, \dots, \beta_m); \quad |\alpha| = \sum_{j=1}^m \alpha_j;$$

$$D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m};$$

D_j is the operator of differentiation with respect to the coordinate x_j; D^0 is the identity operator.

With respect to the N-dimensional vector coefficients a_α we shall assume that the following conditions are fulfilled:

1. For all $x \in \Omega$ and $u \in W_2^{(2)}(\Omega)$, $a_\alpha \in L_2(\Omega)$.
2. The functions $a_\alpha(x; p_0, \dots, p_\beta)$ $(0 \le |\alpha|, |\beta| \le r)$, as functions of all variables p_β, are continuously differentiable functions of these variables when the latter vary in any bounded domain.
3. For any collection of real vectors ξ_β $(0 \le |\beta| \le r)$ and for any $x \in \Omega$ and p_β the inequalities

$$\sum_{0 \le |\alpha|, |\beta| \le r} \frac{\partial a_\alpha}{\partial p_\beta} \xi_\alpha \xi_\beta \ge \frac{\mu_0}{(1+T^2)^s} \sum_{0 \le |\alpha| \le r} \xi_\alpha^2, \tag{2}$$

* In work ([3]), in Theorem 1, in the expression for q the radical sign is missing; one should read $q = \sqrt{1-\alpha/\beta}$

where $\mu_0 > 0$, $0 \leqslant \gamma < 1/2$, and

$$T^2 = \sum_{0 \leqslant |\alpha| \leqslant r} p_\alpha^2.$$

4. For any u and $v \in W_2^{(r)}(\Omega)$ the inequality

$$\sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega \left[a_\alpha(x; u, \dots, D^\beta u) - a_\alpha(x; v, \dots, D^\beta v)\right]^2 dx \leqslant K A(u, v), \tag{3}$$

holds, where K is a positive constant and

$$A(u, v) = \sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega \left[a_\alpha(x; u, \dots, D^\beta u) - a_\alpha(x; v, \dots, D^\beta v)\right](D^\alpha u - D^\alpha v)\, dx. \tag{4}$$

Suppose, for example, that the functions a_α depend only on x and $D^\alpha u$, and, as functions of $D^\alpha u$, are increasing. If, moreover, the Lipschitz condition

$$|a_\alpha(x; D^\alpha u) - a_\alpha(x; D^\alpha v)| \leqslant M |D^\alpha u - D^\alpha v|$$

is satisfied, then it is not difficult to show that the functions a_α satisfy inequality (3). It is also easy to see that inequality (3) will be satisfied by the collection of functions $(r = 1)$

$$a_i(x; p_1, \dots, p_n) = p_i \Big/ \left(1 + \sum_{i=1}^{n} p_i^2\right)^{\gamma/2} \qquad (0 \leqslant \gamma \leqslant 1;\; i = 1, \dots, n). \tag{5}$$

With the aid of (2) and Hölder' s inequality, the inequality

$$A(u, v) \geqslant \frac{\mu_0}{2^\gamma} \frac{\|u - v\|^2_{2(1-\gamma)}}{\left[(m\Omega)^{1/(1-\gamma)} + \|v\|^2_{2(1-\gamma)} + \|u - v\|^2_{2(1-\gamma)}\right]^\gamma} \tag{6}$$

is easily proved, where μ_0 is the constant from inequality (2); $m\Omega$ is the measure of the domain Ω; u and v are any two functions from $W^{(r)}_{2(1-\gamma)}$.

Let us now consider equation (1) with the homogeneous boundary conditions of the first boundary-value problem

$$u|_\Gamma = \partial u/\partial \nu|_\Gamma = \dots = \partial^{r-1} u/\partial \nu^{r-1}|_\Gamma = 0. \tag{7}$$

As usual, by a generalized solution of problem (1), (7) we shall mean a function $u \in \dot{W}^{(r)}_{2(1-\gamma)}$ for which, for all $v \in \dot{W}^{(r)}_{2(1-\gamma)}$, the equality

$$\sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega a_\alpha(x; u, \dots, D^\beta u)\, D^\alpha v\, dx = 0 \tag{8}$$

is satisfied.

Consider the following iterative process for finding a solution of problem (1), (7):

$$\sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega D^\alpha u_{n+1} D^\alpha v\, dx = \sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega D^\alpha u_n D^\alpha v\, dx - \varepsilon \sum_{0 \leqslant |\alpha| \leqslant r} \int_\Omega a_\alpha(x; u_n, \dots, D^\beta u_n)\, D^\alpha v\, dx, \tag{9}$$

where all u_n satisfy the boundary conditions (7); u_0 is any function from $\dot{W}^{(r)}_2$, and ε is a certain sufficiently small positive constant. It is easy to see that this process can be realized and, moreover, that each approximation is an element of $\dot{W}^{(r)}_2$.

Indeed, the expression occurring on the right-hand side of equality (9) is a linear functional with respect to v. This follows from the assumptions on the coefficients a_α. By F. Riesz' s theorem, this functional can be represented in the form

$$\sum_{0\leqslant|\alpha|\leqslant r}\int_\Omega D^\alpha \bar{u}_n D^\alpha v\,dx,$$

where $\bar{u}_n$ is some element of $W_2^{(r)}$. Then from equality (9) it follows that $u_{n+1} = \bar{u}_n$.

Theorem 1. If there exists a generalized solution of problem (1), (7) belonging to the space $W_2^{(r)}$, then the process of successive approximations (9) converges to this solution in the norm $W_{2(1-\gamma)}^{(r)}$ for all sufficiently small $\varepsilon > 0$.

Proof. Denote by u^* the solution of problem (1), (7). Then for any $v \in W_2^{(r)}$ the equality

$$\sum_{0\leq|\alpha|<r}\int_\Omega a_\alpha(x; u^*, \dots, D^\beta u^*) D^\alpha v\,dx = 0, \tag{8'}$$

holds, following directly from equality (8). Put $w_n = u_n - u^*$.

Subtract from both sides of equality (9) the expression

$$\sum_{0\leq|\alpha|\leq r}\int_\Omega D^\alpha u^* D^\alpha v\,dx.$$

Then, taking into account identity $(8')$, we obtain

$$\int_\Omega D^\alpha w_{n+1} D^\alpha v\,dx = -\int_\Omega \left\{D^\alpha w_n - \varepsilon\left[a_\alpha(x; u_n, \dots, D^\beta u_n) - a_\alpha(x; u^*, \dots, D^\beta u^*)\right]\right\} D^\alpha v\,dx,$$

where summation is carried out over repeated indices.

Applying to the expression on the right the inequality

$$\sum\int f_i g_i\,dx \leq \left(\sum\int f_i^2\,dx\right)^{1/2}\left(\sum\int g_i^2\,dx\right)^{1/2},$$

we find

$$\left|\int_\Omega D^\alpha w_{n+1} D^\alpha v\,dx\right| \leq \left\{\|w_n\|_2^2 - 2\varepsilon\sum_{0\leq|\alpha|\leq r}\int_\Omega D^\alpha w_n\left[a_\alpha(x; u_n, \dots, D^\beta u_n) - a_\alpha(x; u^*, \dots, D^\beta u^*)\right] dx\right.$$

$$+\varepsilon^2 \sum_{0\le|\alpha|\le r} \int_\Omega \left[a_\alpha(x; u_n, \dots, D^\beta u_n) - a_\alpha(x; u^*, \dots, D^\beta u^*)\right]^2 dx \Bigg\}^{1/2} \|v\|_2.$$

Substituting $v = w_{n+1}$, we obtain, using expression (4),

$$\|w_{n+1}\|_2^2 \le \|w_n\|_2^2 - 2\varepsilon A(u_n, u^*) + \varepsilon^2 \sum_{0\le|\alpha|\le r} \int_\Omega \left[a_\alpha(x; u_n, \dots, D^\beta u_n) - a_\alpha(x; u^*, \dots, D^\beta u^*)\right]^2 dx.$$

Using inequality (3) and taking $\varepsilon = 1/K$, we obtain

$$\|w_{n+1}\|_2^2 \le \|w_n\|_2^2 - \frac{1}{K} A(u_n, u^*).$$

Applying inequality (6), we find

$$\|w_{n+1}\|_2^2 \le \|w_n\|_2^2 - \frac{\mu_0}{2^\gamma K} \frac{\|w_n\|_{2(1-\gamma)}^2}{\left[(m\Omega)^{1/(1-\gamma)} + \|u^*\|_{2(1-\gamma)}^2 + \|w_n\|_{2(1-\gamma)}^2\right]^\gamma}. \tag{10}$$

From the last inequality it follows that $\|w_k\|_2$ decreases and, consequently, $\|w_n\|_2 \le \|w_0\|_2$. Using the embedding inequality $\|w_n\|_{2(1-\gamma)} \le C\|w_n\|_2$ and the fact that $u^* \in W_{2(1-\gamma)}^{(r)}$, we obtain from inequality (10)

$$\|w_{n+1}\|_2^2 \le \|w_n\|_2^2 - D\|w_n\|_{2(1-\gamma)}^2,$$

where D is a certain constant independent of n.

Summing the last relation from zero to N, we obtain

$$\|w_{N+1}\|_2^2 \leqslant \|w_0\|_2^2 - D\sum_{n=0}^{N} \|w_n\|_{2(1-\gamma)}^2.$$

The last inequality implies that the series $\sum_{n=0}^{N} \|w_n\|_{2(1-\gamma)}^2$ converges, whence the assertion of the theorem follows.

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Received
16 III 1966

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Note: Figure translations are in progress. See original paper for figures.