TOPOLOGICAL EQUIVALENCE OF DYNAMICAL SYSTEMS

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Abstract

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TOPOLOGICAL EQUIVALENCE OF DYNAMICAL SYSTEMS

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In this note theorems are formulated which are a generalization and development of results of the author ([1,2]) and of Hartman ([3,4]), concerning the topological equivalence of the systems

$$\dot{x} = Ax + F(x), \tag{1}$$

$$\dot{y} = Ay, \tag{2}$$

where A is a constant $n \times n$ matrix; x, y, F are vectors of the n-dimensional space L^n.

1°. We introduce the following definitions. A homeomorphism Φ mapping some set $G \subset L^n$ onto some set $M \subset L^n$ in such a way that solutions of system (1) from G are transformed into solutions of system (2) from M and conversely will be called an H-homeomorphism of systems (1) and (2) on the sets G and M, and the systems (1) and (2) will be called homeomorphic on G and M.

If the H-homeomorphism Φ has the form $\Phi(x) = x + \varphi(x)$, where the vector $\varphi(x)$ is bounded in norm ($|\varphi(x)| < \text{const}$) for $x \in G$, then we shall say that Φ has a bounded displacement.

If the displacement φ of the H-homeomorphism Φ for all $x \in G$ satisfies the inequality $|\varphi(x)| \leq ac$, where $a > 0$ is determined by the matrix A, and $c = \sup_{x \in G} |F(x)| < +\infty$, then Φ will be called an A-homeomorphism.

2°. Suppose the following:

a) the matrix A has no eigenvalues with zero real part;

b) the vector $F(x)$ is defined and bounded in L^n: $|F(x)| \leq c$ for any x.

Theorem 1. Suppose that conditions a), b) are fulfilled and, in addition, the requirements are satisfied:

c) $F(x)$ is continuous in L^n and ensures uniqueness of the solution of the Cauchy problem for system (1);

d) the difference of any two solutions of system (1) is unbounded on the t-axis.

Then

1) there exists an A-homeomorphism Φ of systems (1) and (2), mapping L^n onto itself;

2) the homeomorphism Φ is the unique A-homeomorphism of systems (1) and (2) in L^n, and even the unique H-homeomorphism of systems (1) and (2) in L^n with bounded displacement;

3) condition d) is necessary for the existence in L^n of an H-homeomorphism of systems (1) and (2) with bounded displacement;

4) there exists an infinite set of H-homeomorphisms of systems (1) and (2).

Theorem 2. If requirements a) and b) are fulfilled, and the vector $F(x)$ satisfies in L^n a Lipschitz condition with a sufficiently small constant, then assertions 1), 2), and 4) of Theorem 1 are true.

Theorem 3. If condition a) is satisfied, and the vector $F(x)$ satisfies the Lipschitz condition in some bounded domain $G \subset L^n$ with a sufficiently small constant, then there exists an A-homeomorphism of the systems (1) and (2) mapping the domain G onto some domain M of the space L^n.

Theorem 4. If condition a) is fulfilled, and the vector $F(x)$ satisfies the Lipschitz condition in L^n with a sufficiently small constant, then the systems (1) and (2) are homeomorphic in L^n.

3°. Here we shall formulate a theorem on the homeomorphism of abstract dynamical systems. In doing so we shall use the terminology and notation adopted in the book [5].

Let a dynamical system $\{R, f(p,t)\}$ be given, where R is a metric space, p is a point of R, $t \in (-\infty, +\infty)$, and $f(p,t)$ is a one-parameter group of mappings (motions) of R onto itself.

We shall say that the point $f(p,t')$ is an **entry point**, and the instant $t = t'$ the **entry time** of the motion $f(p,t)$ into a certain closed set M, if $f(p,t') \in M$, but for $t < t'$, $f(p,t) \notin M$. Analogously we define the exit point and the exit time $f(p,t)$ from M.

Since motions whose trajectories coincide cannot have different entry points into M (exit points from M), it makes sense to speak of entry points into M (exit points from M) of trajectories.

A closed set M will be called **separating** for the dynamical system $\{R, f\}$ if:

1) every trajectory of the system $\{R, f\}$ intersects M;

2) if not the whole trajectory lies in M, then either the trajectory has an entry point p^* into M and the positive semitrajectory $f(p^*, I^+) \subset M$; or the trajectory has an exit point q^* from M and the negative semitrajectory $f(q^*, I^-) \subset M$; or the trajectory has both an entry point into M and an exit point from M, and the arc of the trajectory enclosed between these points belongs to M;

3) for the motion $f(p, t)$, the entry time into M (exit time from M), if it exists, depends continuously on p.

Obviously, the concept of an H-homeomorphism is also applicable to abstract dynamical systems.

Theorem 5. Suppose that each of the dynamical systems $\{R, f_i\}$, $i = 1, 2$, has a separating set M_i, and suppose that there exists an H-homeomorphism Φ^*, $\Phi^*(M_1) = M_2$, of the systems $\{R, f_1\}$ and $\{R, f_2\}$.

Then one can construct an H-homeomorphism Φ of the systems $\{R, f_1\}$ and $\{R, f_2\}$, mapping R onto itself and coinciding with Φ^* on M_1.

Obviously, the meaning of Theorem 5 is that it asserts the possibility of extending an H-homeomorphism from M_1 to R.

Theorem 4 is obtained from Theorem 2 with the aid of Theorem 5.

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Note: Figure translations are in progress. See original paper for figures.