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Abstract

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ON THE SOLVABILITY OF CERTAIN EQUATIONS IN DENSE SEQUENCES OF INTEGERS

(Presented by Academician A. N. Kolmogorov, 4 IV 1967)

In the preceding paper (^{1}), using a simple combinatorial result of Kleitman (^{4}), we showed that if $a_1 < a_2 < \cdots$ is an infinite sequence of integers such that for infinitely many x the inequality

$$Ax = \sum_{a_i \le x} \frac{1}{a_i} > c_1 \log x / (\log \log x)^{1/2},$$

holds, then the equations $(a_i, a_j) = a_r,\ r < i < j,\ [a_{i_1}, a_{j_1}] = a_{r_1},\ i_1 < j_1 < r_1$, have infinitely many solutions. We also showed that, in a certain sense, this theorem cannot be improved, namely, that the constant c_1 cannot be replaced by an arbitrarily small constant. More precisely, we constructed a sequence satisfying the condition

$$\sum_{a_i \le x} 1 > c_2 x / (\log \log x)^{1/2}, \tag{1}$$

but nevertheless the equation $[a_{i_1}, a_{j_1}] = a_{r_1},\ i_1 < j_1 < r_1$, is not solvable.

In this paper, by $c, c_1, c_2, \ldots$ we shall denote absolute constants; by p, prime numbers; by $P(n)$, the largest, and by $p(n)$, the smallest prime divisor of the number n. The sequence $a_1 < a_2 < \cdots$ will be denoted by A.

We shall say that a sequence $u_1 < u_2 < \cdots$ satisfies property I if the equation $u_{iq} = u_j,\ p(q) > P(u_i)$ has no solutions.

In the present paper we shall show that the equation $(a_i, a_j) = a_r$ behaves quite differently from the equation $[a_i, a_j] = a_r$. We shall prove the following theorem:

Theorem. Let $a_1 < \cdots$ be a sequence of integers for which the equation

$$(a_i, a_j) = a_r, \quad r < i < j, \tag{2}$$

has no solutions. Then

$$\sum \frac{1}{a_i \log a_i} < c. \tag{3}$$

We shall make some preliminary remarks. By partial summation, from the theorem of our paper (²) we easily obtain that, if equation (2) has no solutions, then for every k the equality

$$\liminf_{x\to\infty} \sum_{a_i \le x} \left(\frac{x}{\prod_{r=2}^{k} \log_r x} \right)^{-1} = 0$$

holds (by $\log_r x$ is denoted the r-th iteration of the logarithm).

Thus, relations similar to (1) cannot occur in this case.

A sequence $b_1 < \cdots$ is called **primitive** if there does not exist a number b by which all the remaining terms of the sequence are divisible.

⋯It is well known (³) that for every primitive sequence the inequality

$$\sum \frac{1}{b_i \log b_i} < c_3, \tag{4}$$

holds, and also (see (2)) the equality

$$\lim_{x=\infty} \sum_{b_i \le x} \frac{1}{b_i} \left(\frac{\log x}{(\log\log x)^{1/2}} \right)^{-1} = 0, \tag{5}$$

holds; moreover, this relation cannot be improved.

We prove that if $a_1 < a_2 < \cdots$ is an infinite sequence for which equation (2) is not solvable, then

$$\lim_{x\to\infty} \sum_{a_i \le x} \frac{1}{a_i} \left(\frac{\log x}{(\log\log x)^{1/2}} \right)^{-1} = 0. \tag{6}$$

The proof of equality (6) is quite complicated, and we shall return to it later. Relations (3), (4), (5), and (6) give rise to the following question. Let $b_1 < b_2 < \cdots$ be an infinite primitive sequence. Does there exist a constant $c > 0$ and a sequence $a_1 < \cdots$ for which equation (2) is not solvable and $a_n \le b_n^c$? We cannot answer this question.

We now pass to the proof of the theorem. We use the following lemma, due to Alexander.

Lemma 1. Let $a_1 < a_2 < \cdots$ be a sequence satisfying property I. Then

$$\sum_i \frac{1}{u_i \log u_i} < c_4. \tag{7}$$

If $u_i \nmid u_j$ (i.e., if the sequence $u_1 < u_2 < \cdots$ is primitive), then inequality (7) was proved in (^{3}). The proof of Lemma 1 is similar to the proof given in (^{3}), but for completeness we shall give a sketch of it here. It is not hard to see that condition I means (see (^{3})) that $u_{iq} = u_j q'$, $p(q) > P(u_i)$, $p(q') > P(u_j)$.

Using the sieve of Eratosthenes, we conclude that the number of integers $u_{iq} \leq x$, $p(q) > P(u_i)$, is greater than

$$\prod_{p \leq P(u_i)} \left(1 - \frac{1}{p}\right) - 2^{u_i}. \tag{8}$$

From relation (8) we easily obtain the inequality

$$\sum_i \prod_{p \leq P(u_i)} \left(1 - \frac{1}{p}\right) / u_i \leq 1, \tag{9}$$

from which, with the aid of Mertens' theorem

$$\prod_{p<y} \left(1 - \frac{1}{p}\right) < c/\log y,$$

the proof of our lemma follows.

We now define a subsequence $A(a_i)$ of the sequence A as follows: a_j is contained in $A(a_i)$ if the number a_i is the greatest of those a for which the equation $a_j = a_{iq}$, $p(q) > P(a_i)$, is solvable. Let A' be the subsequence of the sequence A which is not contained in any subsequence $A(a_i)$. It is clear that

$$A = A' \cup \bigcup_{i=1}^{\infty} A(a_i).$$

Thus,

$$\sum_k \frac{1}{a_k \log a_k} = \sum_{a_k \in A'} \frac{1}{a_k \log a_k} + \sum_{i=1}^{\infty} \sum_{a_k \in A(a_i)} \frac{1}{a_k \log a_k}. \tag{10}$$

Obviously, the subsequence A' satisfies property I. Thus, by Lemma 1, the inequality

$$\sum_{a_k \in A'} \frac{1}{a_k \log a_k} < c_4. \tag{11}$$

holds.

We now prove Lemma 2.

Lemma 2.

$$\sum_{a_k \in A(a_i)} \frac{1}{a_k \log a_k} < \frac{c_5}{a_i P(a_i)^{1/2}}.$$

It is easy to see (the $q_1 < q_2 < \cdots$ run through the set of all primes) that

$$= \sum \frac{1}{n(P(n))^{1/2}} = \sum_{m=1}^{\infty} \frac{1}{q_m^{3/2}} \prod_{i=1}^{m} \left(1 + \frac{1}{q_i}\right) < \sum_{m=1}^{\infty} \frac{c \log q_m}{q_m^{3/2}} < \infty.$$

Our Theorem 1 therefore follows immediately from (10), (11), and Lemma 2. To complete the proof it remains only to establish Lemma 2. Let $a_i q_r^{(i)}$, $r = 1, \dots$, $p(q_r^{(i)}) > P(a_i)$, be the integers of the subsequence $A(a_i)$. It is clear that the sequence $q_r^{(i)}$ satisfies property I. If this is not so and $q_{r_2}^{(i)} / q_{r_1}^{(i)}$ is an integer satisfying the inequality $p(q_{r_2}^{(i)} / q_{r_1}^{(i)}) > P(q_{r_1}^{(i)})$, then $a_i q_{r_2}^{(i)}$ (which belongs to the subsequence $A(a_i)$) can be written in the form $a_l q$, $p(q) > P(a_l)$, $a_l = a_i q_r^{(i)}$, $q_{r_2}^{(i)} / q_{r_1}^{(i)} = q$, which contradicts the maximality of a_i.

We now show that there do not exist two relatively prime numbers $q_r^{(i)}$. To see this, we first use the fact that equation (2) has no solutions. Namely, assuming that $(q_{r_1}^{(i)}, q_{r_2}^{(i)}) = 1$, we obtain $(a_i q_{r_1}^{(i)}, a_i q_{r_2}^{(i)}) = a_i$. In other words, equation (2) has a solution, contrary to our assumption.

Lemma 3. Let the sequence $q_1 < \cdots$ satisfy property I, $(q_i, q_j) \neq 1$, and $p(q_i) > t$. Then

$$\sum_i \frac{1}{q_i \log q_i} \ll \frac{c_5}{t^{1/2}}.$$

Lemma 2 immediately follows from Lemma 3, since

$$\sum_{a_k \in A(a_i)} \frac{1}{a_k \log a_k} = \sum_r \frac{1}{a_i q_r^{(i)} \log a_i q_r^{(i)}} \ll \frac{1}{a_i} \sum_r \frac{1}{q_r^{(i)} \log q_r^{(i)}} < \frac{c_5}{a_i p(a_i)^{1/2}}.$$

Thus, it remains only for us to prove Lemma 3. It is quite likely that Lemma 3 is not the best possible, and that the expression $c_5/t^{1/2}$ can be replaced by c_6/t.

For the proof of Lemma 3 we first assume that there exists an i for which

$$\sum_{p/q_i} \frac{1}{p} \ll \frac{1}{t^{1/2}}. \tag{12}$$

Since there do not exist two relatively prime numbers q_r, every q^r must be divisible by at least some p, where $p \mid q_i$. Therefore

$$\sum_r \frac{1}{q_r \log q_r} \ll \sum_{p/q_i} \frac{1}{p} \sum{}' \frac{1}{q_r/p \log q_r}, \tag{13}$$

where the prime indicates that the summation extends over all q such that $p \mid q$. Obviously, the sequence q_r/p satisfies property I (except that one of the numbers q_r/p may be equal to one).

Thus, by Lemma 1,

$$\sum{}' \frac{1}{q_r \log q_r} < 1 + c_3. \tag{14}$$

From inequalities (12), (13), and (14) we obtain that

$$\sum_r \frac{1}{q_r \log q_r} < (1 + c_3) \sum_{p/q_i} \frac{1}{p} \leq \frac{1 + c_3}{t^{1/2}}.$$

This proves the lemma.

To complete our proof, suppose now that inequality (12) is false for q_r. Let l be an integer and let $x > x_0(l)$ be large. Consider the integers not exceeding x of the form $q_r(t)$, where all prime factors of t are greater than q_r. Since the sequence q_r satisfies property I, we obtain, as in Lemma 1, that the integers

$$q_r m, \quad r = 1, 2, \ldots, l, \quad m < x/q_r, \tag{15}$$

are distinct. Denote by $u_1, u_2, \ldots, u_s$ the numbers of the form (15). By Mertens' theorem and the sieve of Eratosthenes, we obtain

$$s = (1 + O(1)) \sum_{r=1}^{l} \frac{x}{q_r} \prod_{p=P(q_r)} \left(1 - \frac{1}{p}\right) > Cx \left(\sum_r \frac{1}{q_r \log q_r}\right) + O(x). \tag{16}$$

Obviously, all prime factors of u are greater than t, and since inequality (12) is false, we have

$$\sum_{p/u_i} \frac{1}{p} > \frac{1}{t^{1/2}}.$$

Thus, on the one hand,

$$\sum_{i=1}^{s} \sum_{p/u_i} \frac{1}{p} > \frac{s}{t^{1/2}}, \tag{17}$$

and on the other,

$$\sum_{i=1}^{s} \sum_{p/u_i} \frac{1}{p} < \sum_{u=1}^{x} \sum_{\substack{p/u \\ p>t}} \frac{1}{p} < \sum_{p>t} \frac{x}{p^2} < \frac{x}{t}. \tag{18}$$

Therefore, from inequalities (17) and (18) we obtain the inequality

$$s < x/t^{1/2}. \tag{19}$$

Thus inequalities (16) and (19) give the inequality

$$\sum_{r=1}^{l} \frac{1}{q_r \log q_r} < c_5 t^{1/2}, \tag{20}$$

and since the last inequality is valid for every l, the proof of Lemma 3, and consequently of the theorem, is complete.

Our proof does not use Kleitman’ s combinatorial result [4]. We do not know how to deal with the equation $[a_i, a_j] = a_r$ without using Kleitman’ s result.

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