ON THE EXTENSION OF MAPPINGS OF TOPOLOGICAL SPACES

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.831

MATHEMATICS

N. V. VELICHKO

ON THE EXTENSION OF MAPPINGS OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov, February 27, 1967)

The paper is devoted to questions concerning the extension of continuous and θ-continuous (see (1)) mappings of topological Hausdorff spaces to their Hausdorff (mainly H-closed) extensions. A space X is called **completely separated** if any two distinct points of it have disjoint closed neighborhoods. A set P in a space X is called δ-closed if, for every point $x \notin P$, there is a neighborhood Ox such that $\langle [Ox] \rangle \cap P = \emptyset$ (everywhere here $[A]$ denotes the closure, and $\langle A \rangle$ the open kernel of the set A) (see (12)). An extension ρX of a space X is called **paracombinatorial** if, for every pair of open disjoint sets in X, the intersection of their closures in ρX lies in X.

Lemma 1. *A Hausdorff extension αX of a Hausdorff space X is H-closed if and only if, for every centered system of sets $\{H_\alpha\}$ open in X, we have*

$$\bigcap_\alpha [H_\alpha]_{\alpha X} \neq \emptyset.$$

Theorem 1. *Every perfect continuous mapping f of a Hausdorff space X onto a Hausdorff space Y extends, with preservation of continuity, to a mapping $\hat{f}$ of the Katětov extension τX onto an arbitrary H-closed extension cY of the space Y in such a way that*

$$\hat{f}(\tau X \setminus X) = cY \setminus Y.$$

Proof. Let $\xi \in \tau X \setminus X$, let $\{H'_\alpha\}_{\alpha \in I}$ be the system of all its open neighborhoods in τX, and let $\{H_\alpha\} = \{H'_\alpha \cap X\}$. Then

$$\bigcap_\alpha [H_\alpha]_X = \emptyset.$$

In each set H_α choose a point x_α. Let $B_\alpha = f^{-1} f x_\alpha$. The set B_α is bicompact. Denote by $\{O_{\gamma_\alpha} B_\alpha\}_{\gamma_\alpha \in \Gamma_\alpha}$ the system of all open neighborhoods in X of the bicompactum B_α. Construct the systems $\{O_{\gamma_\alpha} B_\alpha\}$ for each $\alpha \in I$. Consider the system

$$\{T_{\gamma_\alpha, \alpha}\} = \{H_\alpha \cup O_{\gamma_\alpha} B_\alpha\},$$

where γ_α runs through Γ_α, and α runs through I. We shall prove that

$$\bigcap_{\gamma_\alpha\in\Gamma_\alpha,\ \alpha\in I}[T_{\gamma_\alpha,\alpha}]_{\tau X} = \{\xi\}.$$

Let a point $\eta \neq \xi$ be arbitrary, and let $O\eta$ and $O\xi$ be disjoint (open) neighborhoods of the points η and ξ in τX. Let $O\xi \cap X = H_\alpha$. Since B_α is bicompact, the point η has a neighborhood $V\eta$ such that $[V\eta] \cap B_\alpha = \emptyset$. Then

$$V\eta \cap O\eta \cap T_{\gamma_\alpha,\alpha} = \emptyset,$$

where

$$T_{\gamma_\alpha,\alpha} = H_\alpha \cup (X \setminus [V\eta \cap X]_X).$$

Consequently,

$$\eta \notin \bigcap[T_{\gamma_\alpha,\alpha}]_{\tau X} \quad \text{and} \quad \bigcap[T_{\gamma_\alpha,\alpha}]_{\tau X} = \{\xi\}.$$

The set $X \setminus T_{\gamma_\alpha,\alpha}$ is closed in X; therefore the closed set

$$S_{\gamma_\alpha,\alpha} = f(X \setminus T_{\gamma_\alpha,\alpha})$$

does not cover Y, since $fx_\alpha \notin S_{\gamma_\alpha,\alpha}$. Put

$$L_{\gamma_\alpha,\alpha} = Y \setminus S_{\gamma_\alpha,\alpha}.$$

The family

$$t = \{L_{\gamma_\alpha,\alpha}\}_{\gamma_\alpha\in\Gamma_\alpha,\ \alpha\in I}$$

of open sets in Y is centered. Indeed, consider an arbitrary finite collection of sets from t:

$$\{L_{\gamma_{\alpha_i},\alpha_i}\}, \quad i = 1, 2, \dots, n.$$

Let

$$H_\beta = \bigcap_{i=1}^{n} H_{\alpha_i}.$$

Then

$$fx_\beta \in \bigcap_{i=1}^{n} L_{\gamma_{\alpha_i},\alpha_i} \quad \text{and} \quad \bigcap_{i=1}^{n} L_{\gamma_{\alpha_i},\alpha_i} \neq \emptyset.$$

From the centeredness of the system $\{L_{\gamma_\alpha,\alpha}\}$ it follows that

$$\bigcap_{\gamma_\alpha,\alpha}[L_{\gamma_\alpha,\alpha}]_{cY} = B \neq \emptyset$$

(Lemma 1). We shall prove,

that the set B reduces to a single point y. Suppose that B contains two points y_1 and y_2. Let Oy_1 and Oy_2 be their disjoint open neighborhoods. Then

$$f^{-1}(Oy_1 \cap Y) \cap f^{-1}(Oy_2 \cap Y) \neq \emptyset,$$

but

$$T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_1 \cap Y) \neq \emptyset \quad \text{and} \quad T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_2 \cap Y) \neq \emptyset$$

for arbitrary γ_α and α.

We shall prove that: (A) for any open neighborhood Oy of an arbitrary point $y \in B$ we have

$$\xi \in [f^{-1}(Oy \cap Y)]_{\tau X}.$$

Indeed, suppose the contrary: assume that the point ξ has an open neighborhood $O\xi$ not intersecting $f^{-1}Oy$. Let $O\xi \cap X = H_\alpha$. Then $fx_\alpha \neq y$. Let Vy and Vfx_α be disjoint open neighborhoods of the points y and fx_α in cY. Then, for

$$T_{\gamma_\alpha,\alpha} = H_\alpha \cup f^{-1}(Vfx_\alpha)$$

and

$$Wy = Vy \cap Oy,$$

we would have

$$Wy \cap L_{\gamma_\alpha,\alpha} = \emptyset,$$

which is impossible. Thus (A) is proved. Consequently,

$$\xi \in [T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_1 \cap Y)]_{\tau X} \cap [T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_2 \cap Y)]_{\tau X},$$

where $T_{\gamma_\alpha,\alpha}$ is chosen arbitrarily. But the sets

$$T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_1 \cap Y) = M_1$$

and

$$M_2 = T_{\gamma_\alpha,\alpha} \cap f^{-1}(Oy_2 \cap Y)$$

are open in X and do not intersect. Therefore (since τX is a paracombinatorial extension)

$$[M_1]_{\tau X} \cap [M_2]_{\tau X} \cap (\tau X \setminus X) = \emptyset.$$

We have arrived at a contradiction. Hence $B = \{y\}$.

Put $\hat{f}\xi = y$, and $\hat{f}x = fx$ for points $x \in X$. We obtain a mapping $\hat{f}$ of the space τX into the space cY, which extends f, and

$$\hat{f}(\tau X \setminus X) \subseteq cY \setminus Y.$$

The latter inclusion follows from the fact that the point $\hat{f}\xi = y \in cY \setminus Y$, for for any point $\bar{y} \in Y$ the set $B\bar{y} = f^{-1}\bar{y}$ and the point $\xi \in \tau X \setminus X$ have disjoint open neighborhoods $OB\bar{y}$ and $O\xi$, and then, if $O\xi \cap X = H_\alpha$, we have

$$\bar{y} \notin L_{\gamma_\alpha,\alpha}$$

for some γ_α.

It is clear that the mapping $\hat{f}$ is continuous at each point $x \in X$. Let $\xi \in \tau X \setminus X$, $y = \hat{f}\xi$, and let Oy be an arbitrary open neighborhood of the point y in cY, $T = Oy \cap Y$. Then

$$\xi \in [f^{-1}T]_{\tau X}$$

(see (A)), and since $f^{-1}T$ is open in X, it follows that

$$\{\xi\} \cup f^{-1}T = O\xi$$

is a neighborhood of the point ξ such that

$$\hat{f}\, O\xi \subseteq Oy.$$

Consequently, $\hat{f}$ is continuous on τX, and since

$$\hat{f}\tau X \supseteq Y,$$

we have

$$\hat{f}\tau X = cY,$$

and

$$\hat{f}(\tau X \setminus X) = cY \setminus Y.$$

The theorem is proved.

Corollary. The Katětov extension τX of a Hausdorff space X is mapped continuously and fixing the points of X onto every H-closed extension cX of the space X, and the remainder $\tau X \setminus X$ is mapped onto the remainder $cX \setminus X$.

Theorem 2. If f is a perfect continuous mapping of a Hausdorff space X onto an H-closed space Y, then X is also H-closed.

Proof. The mapping $f : X \to Y$ can be extended, preserving continuity, to a mapping

$$\hat{f} : \tau X \to Y$$

(Theorem 1), but a perfect mapping is absolutely closed (see (2)), which entails the equality

$$\tau X = X.$$

Consequently, the space X is H-closed. The theorem is proved.

Definition. A mapping f of a space X onto a space Y is called δ-continuous if, for every point $x \in X$ and every neighborhood Ou of the point $y = fx \in Y$, there is a neighborhood Ox of the point x such that

$$fOx \subseteq \langle [Oy] \rangle.$$

Lemma 2. A mapping $f : X \to Y$ is δ-continuous if and only if, for every set H open in Y, the set

$$f^{-1}(\langle [H] \rangle)$$

is open in X.

Theorem 3. A θ-continuous mapping f of a Hausdorff space X onto a Hausdorff space Y is θ-continuous in each of the following cases: (a) the space X is H-closed and completely regular; (b) the mapping f is open.

Proof. (a). The space Y is H-closed and completely regular. Let x be an arbitrary point of X, and let Oy be an arbitrary neighborhood of the point $y = fx$. We shall prove that the point y has a neighborhood Vy such that

$$[Vy] \subseteq \langle [Oy] \rangle.$$

The set

$$S = Y \setminus \langle [Oy] \rangle = [Y \setminus [Oy]]$$

is H-closed as the closure of an open set. For each point $z \in S$

and the point y have open neighborhoods Oz and Ozy such that $[Oz] \cap [Ozy] = \emptyset$. There is a finite number of points z_i, $i = 1, 2, \dots, n$, such that $\bigcup_{i=1}^{n} [Oz_i] \supseteq S$. Put $Vy = \bigcap_{i=1}^{n} Oz_i y$. We have $[Vy] \cap S = \emptyset$ (for $[Oz_i y] \cap [Oz_i] = \emptyset$), whence $[Vy] \subseteq \langle [Oy] \rangle$. Let Ox be a neighborhood of the point x such that $fOx \subseteq [Vy]$. Then $fOx \subseteq \langle [Oy] \rangle$. (a) is proved.

(b). The mapping $f : X \to Y$ is open. Let H be an arbitrary open set in Y. Since for each point $z \in f^{-1}(\langle [H] \rangle) = K$ there is a neighborhood Oz such that $fOz \subseteq [H]$, the set $L = \bigcup_{z \in K} Oz$ will be an open set such that $fL = [H]$, $fL \subseteq \langle [H] \rangle$, $fL \supseteq \langle [H] \rangle$; consequently, $fL = \langle [H] \rangle$, $L = f^{-1}(\langle [H] \rangle)$, and the mapping f is δ-continuous (Lemma 2). (b) is established. The theorem is proved.

Theorem 4. *Every open θ-continuous mapping f of a Hausdorff space X onto a Hausdorff space Y can be extended, with preservation of θ-continuity, to a mapping $\hat{f}$ of any paracombinatorial (not necessarily H-closed) extension pX into any H-closed extension cY of the space Y, in such a way that $\hat{f}(pX \setminus X) \subseteq cY \setminus Y$.*

Proof. Let $p \in pX \setminus X$ be arbitrary, and let $\{H'_\alpha\}_{\alpha \in I}$ be the system of all its open neighborhoods in pX, $\{H_\alpha\} = \{H'_\alpha \cap X\}$. The family $\{fH_\alpha\}$ of open sets in Y is centered; therefore $B = \bigcap_\alpha [fH_\alpha]_{cY} \neq \emptyset$ (Lemma 1). The set B reduces to a single point η. Suppose the contrary—that B contains two points η_1 and η_2. Let $O\eta_1$ and $O\eta_2$ be disjoint open neighborhoods of the points η_1 and η_2. In view of the openness of f, the mapping f is δ-continuous (Theorem 3). Then the sets $L_1 = f^{-1}(\langle [O\eta_1 \cap Y] \rangle)$ and $L_2 = f^{-1}(\langle [O\eta_2 \cap Y] \rangle)$ are open in X and do not intersect. But $[L_1]_{pX} \cap [L_2]_{pX} = K \ni p$, which contradicts the paracombinatoriality of the extension pX. Hence $B = \{\eta\}$.

Put $\hat{f}p = \eta$, $\hat{f}x = fx$ for points $x \in X$. We shall prove that the mapping $\hat{f}$, which is an extension of f, is θ-continuous.

(M). Let the set A be open in X, and let $z \in [A]_{pX} \setminus [A]_X$. Then $\xi = \hat{f}z \in [fA]_{cY}$. Indeed, if the point ξ had an open neighborhood $O\xi$ not intersecting

fA, then the open sets $S = f^{-1}(\langle[O\xi \cap Y]\rangle)$ and A in X would not intersect, but $[S]_{pX} \cap [A]_{pX} = L \ni z \in pX \setminus X$, which contradicts the paracombinatoriality of pX.

Let the point $x \in X$ be arbitrary; $y = \hat{f}x = fx$; let Oy be an arbitrary open neighborhood of the point y; let Ox be an open neighborhood of the point x in X such that $fOx \subseteq \langle[Oy \cap Y]\rangle$; let $\widehat{Ox}$ be the largest open set in pX that cuts out Ox in X. If $z \in \widehat{Ox} \setminus Ox$, then $z \in [Ox]_{pX}$, whence $\hat{f}z \in [fOx]_{cY} \subseteq [Oy]_{cY}$ (see (M)); consequently, $\hat{f}\widehat{Ox} \subseteq [Oy]_{cY}$, i.e. the mapping $\hat{f}$ is θ-continuous at the point $x \in X$.

Now let $p \in pX \setminus X$, $\eta = \hat{f}p \in cY$, and let $O\eta$ be an arbitrary open neighborhood of the point η in cY. The set $K = f^{-1}(\langle[O\eta \cap Y]\rangle_Y)$ is open in X, and $p \in [K]_{pX}$. Note that $[K]_{pX} \cup [X \setminus [K]]_{pX} = pX$, and, by the paracombinatoriality of the extension pX, the point $p \notin [X \setminus [K]]_{pX}$; therefore the point p has a neighborhood Op such that $Op \subseteq [K]_{pX}$. Let the point $a \in [K]_{pX} \setminus X$ be arbitrary. Then $b = \hat{f}a \in [O\eta]_{cY}$ (see (M)); consequently, $fOp \subseteq [O\eta]_{cY}$, i.e. the mapping $\hat{f}$ is θ-continuous at the point $p \in pX \setminus X$ and, thus, θ-continuous on pX. It is not hard to verify that if $p \in pX \setminus X$, then the point $\eta = \hat{f}p \in cY \setminus Y$, i.e. $\hat{f}(pX \setminus X) \subseteq cY \setminus Y$. The proof is complete.

Theorem 5. *In order that a θ-continuous mapping f of an everywhere dense set S in a space X into an H-closed completely Hausdorff space Y can be extended to all of X with preservation of θ-*

continuity, it is necessary and sufficient that, for any two disjoint canonically closed sets B_1 and B_2 in Y, the preimages $f^{-1}B_1$ and $f^{-1}B_2$ have disjoint closures in X.

Proof of necessity. The θ-continuous mapping $f : S \to Y$ extends to all of X. The mapping f is δ-continuous (Theorem 3). Let B_1 and B_2 be canonically closed disjoint sets in Y. B_1 and B_2 are δ-closed as closures of open sets (Lemma 2 from ([3])). Suppose, for example, that the point $\xi \in [f^{-1}B_1]_X$. Then $f\xi \in [B_1]_\delta = B_1$, by the δ-continuity of f; therefore $\xi \notin [f^{-1}B_2]_X$ and

$$[f^{-1}B_1] \cap [f^{-1}B_2]_X = \emptyset.$$

Proof of sufficiency. Let a point $x \in X$ be arbitrary, and let $\mathfrak{F}$ be the trace on S of the filter of neighborhoods Bx of the point x. $f(\mathfrak{F})$ is a filter base in Y, and since Y is H-closed, $f(\mathfrak{F})$ has points of δ-adherence (Theorem 2 from ([3])). We prove that the set

$$L = \bigcap [fV_\alpha]_\delta$$

reduces to a single point y (where $V_\alpha = S \cap W_\alpha$, and W_α ranges over Bx). Let $y_1 \neq y_2$, $y_1, y_2 \in L$. Choose open neighborhoods Oy_1 and Oy_2 of the points y_1 and y_2 such that $[Oy_1] \cap [Oy_2] = \emptyset$. The sets $[Oy_1]$ and $[Oy_2]$ are canonically closed; hence

$$[f^{-1}([Oy_1])]_X \cap [f^{-1}([Oy_2])]_X = \emptyset.$$

But $V_\alpha \cap f^{-1}([Oy_1]) \neq \emptyset$ and $V_\alpha \cap f^{-1}([Oy_2]) \neq \emptyset$ for every V_α; consequently the point

$$x \in [f^{-1}([Oy_1])]_X \cap [f^{-1}([Oy_2])].$$

The contradiction obtained shows that the set L consists of a single point y. By Lemma 2 from ([4]), $f(\mathfrak{F})$ δ-converges to the point y (i.e., contains the nuclei of the closures of all neighborhoods of the point y).

Put $\hat{f}x = y$. It is clear that $\hat{f}x = fx$ on the set S. We prove that $\hat{f}$ is a θ-continuous extension of the mapping f to X. Consider an arbitrary open neighborhood Oy of the point $y = \hat{f}x$. There exists an open neighborhood W_α of the point x such that $fV_\alpha \subseteq [Oy]$, where $V_\alpha = S \cap W_\alpha$. For an arbitrary point $z \in W_\alpha$ we have

$$\hat{f}z \in [fV_\alpha]_\delta = [[Oy]]_\delta = [Oy]$$

(for the image of the trace on S of the filter of neighborhoods of the point z δ-converges to the point $\hat{f}z$), i.e. $\hat{f}W_\alpha \subseteq [Oy]$. The theorem is proved.

Corollary 1 (A. D. Taimanov' s theorem). *In order that a continuous mapping f from an everywhere dense subset S of a space X into a bicompact space Y be extendable to all of X with preservation of continuity, it is necessary and sufficient that, for any two canonically closed disjoint sets B_1 and B_2 in Y, the preimages $f^{-1}B_1$ and $f^{-1}B_2$ have disjoint closures in X.*

Corollary 2. *Every continuous perfect mapping of an everywhere dense subset S of a Hausdorff space X into an H-closed completely nonseparable space Y extends to a θ-continuous mapping of the space X into the space Y.*

Ural State University
named after A. M. Gorky

Received
1 II 1967

REFERENCES

[1] S. Fomin, *Ann. Math.*, **44**, No. 5, 471 (1943).
[2] A. V. Arhangel' skii, *Tr. Mosk. matem. obshch.*, **13**, 5 (1964).
[3] N. V. Velichko, *Matem. sborn.*, **70** (112), 1, 98 (1966).
[4] N. V. Velichko, *Sibirsk. matem. zhurn.*, **6**, 1, 64 (1965).

Note: Figure translations are in progress. See original paper for figures.