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Abstract

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ON THE INFINITE DIFFERENTIABILITY AND ANALYTICITY OF SOLUTIONS, DECREASING AT INFINITY, OF EQUATIONS WITH CONSTANT COEFFICIENTS

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I. G. Petrovskii ([1]) established necessary and sufficient conditions for the analyticity of all solutions of a linear partial differential equation with constant coefficients (the ellipticity condition). L. Hörmander ([2]) found necessary and sufficient conditions for all solutions of an equation with constant coefficients to be infinitely differentiable functions (the hypoellipticity condition).

In the present note we consider the question of the infinite differentiability and analyticity of those solutions of partial differential equations with constant coefficients which, together with some of their derivatives, decrease in a definite way at infinity.

Let E_n be the n-dimensional space of points $x = (x_1, \dots, x_n)$ with real coordinates; let $k = (k_1, \dots, k_n)$, where $k_1, \dots, k_n$ are nonnegative integers. Put $D^k = D_1^{k_1} \cdots D_n^{k_n}$, where $D_p = \partial/\partial x_p$, $p = 1, \dots, n$. We consider the equation

$$Pu \equiv \sum_{k \in K} a_k D^k u = 0, \tag{1}$$

where $a_k = a_{k_1,\dots,k_n}$ are constant real coefficients, and K is a certain bounded set of points with integer nonnegative coordinates.

Denote by $\widehat{K}$ the minimal convex polyhedron containing the set K, by K^* the minimal convex polyhedron containing the set K and the point $(0, \dots, 0)$, and by $k^{(r)}$, $r = 0, \dots, M$, the vertices of the polyhedron $\widehat{K}$, with $k^{(0)} = (0, \dots, 0)$. We shall also use the notation $[k/2] = ([k_1/2], \dots, [k_n/2])$, $|k| = k_1 + \dots + k_n$.

Definition. Let Ω be some domain in E_n. We shall call a function $u(x)$ a **generalized solution of equation** (1) in the domain Ω, if:

1) $D^{k^{(r)}-[k^{(r)}/2]}u \in L_2^{\text{loc}}(\Omega)$, $r = 0, \dots, M$, i.e. $D^{k^{(r)}-[k^{(r)}/2]}u \in L_2(\Omega')$ for every compact $\Omega' \subset \Omega$ (the derivatives are understood in the generalized sense; $D^0 u \equiv u$);

2) For every function $v \in C_0^\infty(\Omega)$

$$\sum_{k\in K} (-1)^{|[k/2]|} \int_\Omega a_k D^{k-[k/2]}u \cdot D^{[k/2]}v \, dx = 0.$$

All classical solutions are generalized in the sense of the definition formulated above.

Equation (1) will be considered in domains of the form

$$\Omega = \Omega_s \times E_{n-s}, \qquad s = 0, 1, \dots, n, \tag{2}$$

where Ω_s is some domain in $E_s = \{-\infty < x_p < \infty,\ p = 1, \dots, s\}$, and

$$E_{n-s} = \{-\infty < x_p < \infty,\ p = s+1, \dots, n\}.$$

If $s = 0$, then in notation (2) E_n is meant; if $s = n$, then an arbitrary domain $\Omega_n \subset E_n$ is meant.

In what follows we shall consider only those generalized solutions of equation (1) which satisfy the condition

$$D^{k(r)-[k(r)/2]}u \in L^{s\text{-loc}}(\Omega_s \times E_{n-s}), \tag{3}$$

i.e. $D^{k(r)-[k(r)/2]}u \in L_2(\Omega'_s \times E_{n-s})$ for every compact $\Omega'_s \subset \Omega_s$, $r = 0, \dots, M$.

The closed faces of the polyhedron $\hat{K}$ of dimension t, $0 \le t \le n-1$, which are simultaneously faces of the polyhedron $\hat{K}$ and do not lie in coordinate planes, will be denoted by $\hat{K}_r^t$, $r = 1, \dots, M_t$; in addition, put $K_r^t = K \cap \hat{K}_r^t$ and

$$K^{n-1} = \bigcap_{r=1}^{M_{n-1}} K_r^{n-1}.$$

Theorem 1. Let $1 \le s \le n$, and let the following conditions be fulfilled:

1) on each of the coordinate axes k_p, $p = 1, \dots, n$, there is at least one point of the set K;

2) if $s = 1$, then for $p = 2, \dots, n$, and if $2 \le s \le n$, then for $p = 1, \dots, n$, the point $k = (k_1, \dots, k_n) \in K$ having maximal coordinate k_p is unique and lies on the axis k_p;

3) for every compact $\Omega_s' \subset \Omega_s$ and for every generalized solution w of equation (1) on $\Omega_s' \times E_{n-s}$ satisfying the condition

$$\sum_{r=0}^{M} \left\| D^{k(r)-[k(r)/2]} w \right\|_{L_2(\Omega_s' \times E_{n-s})} < \infty,$$

the inequality

$$\sum_{r=0}^{M} \left\| D^{k(r)-[k(r)/2]} w \right\|'_{L_2[(\Omega_s')_\delta \times E_{n-s}]} \ll c_\delta \sum_{k \in K} \sum_{\substack{j=(j_1,\dots,j_s,0,\dots,0) \\ 0 \le j < k-[k/2] \\ |j|>0, \text{ if } k \in K^{n-1}}} \left\| D^{k-[k/2]-j} w \right\|'_{L_2[\Omega_s' \times E_{n-s}]}, \tag{4}$$

holds, where $(\Omega_s')_\delta$ is the set of points $x \in \Omega_s'$ whose distance from the boundary of the compact Ω_s' is greater than δ, and c_δ is a certain constant independent of w.

Then all generalized solutions u of equation (1) in the domain $\Omega_s \times E_{n-s}$ satisfying condition (3) are, after modification on a set of measure zero, infinitely differentiable functions; moreover, for every $l = (l_1, \dots, l_n)$ (l_p are natural numbers, $p = 1, \dots, n$) and for every compact $\Omega_s' \subset \Omega_s$,

$$\left\| D^l u \right\|_{L_2[(\Omega_s')_\delta \times E_{n-s}]} \le c_\delta^{(l)} \sum_{r=0}^{M} \left\| D^{k(r)-[k(r)/2]} u \right\|'_{L_2[\Omega_s' \times E_{n-s}]}, \tag{5}$$

where $c_\delta^{(l)}$ is a certain constant independent of u.

In the proof of this theorem, the fractional spaces $W_2^{\vec{r}}$ of Aronszajn–Slobodetskii are used essentially. In particular, an important role is played by L. N. Slobodetskii' s theorem on the estimate of mixed derivatives ([3]).

We shall call the differential operator P **nondegenerate** if, for arbitrary t and r, $t = 0, \dots, n-1$, $r = 1, \dots, M_t$, the polynomial

$$\operatorname{Re} \sum_{k \in K_r^t} a_k (i\xi)^k$$

does not change sign and can vanish only when ξ belongs to one of the coordinate planes ($\xi = (\xi_1, \dots, \xi_n)$, ξ_p real, $p = 1, \dots, n$). The above-described

the class of nondegenerate differential operators is a special case of the class of differential operators considered by V. P. Mikhailov ([4]).

Theorem 2. *Let $1 \leqslant s \leqslant n$, and let P be a nondegenerate differential operator, and suppose that conditions 1) and 2) of Theorem 1 are satisfied.*

Then, for generalized solutions u of equation (1) on $\Omega'_s \times E_{n-s}$, where Ω'_s is some compact subset of E_s, satisfying the condition

$$\sum_{r=0}^{M} \|D^{k^{(r)}/2}u\|_{L_2(\Omega'_s\times E_{n-s})} < \infty,$$

the inequality

$$\sum_{r=0}^{M} \|D^{k^{(r)}/2}u\|_{L_2[(\Omega'_s)_\delta\times E_{n-s}]} \leqslant c\left\{\sum_{k\in K-K^{n-1}} \|D^{k-[k/2]}u\|_{L_2[\Omega'_s\times E_{n-s}]}+\right.$$

$$\left.+\sum_{q=1}^{q^*}\frac{1}{\delta^q}\sum_{k\in K}\sum_{\substack{j=(j_1,\dots,j_s,0,\dots,0)\\ 0\leqslant j\leqslant k-[k/2]\\ |j|=q}} \|D^{k-[k/2]-j}u\|_{L_2[\Omega'_s\times E_{n-s}]}\right\}, \qquad (6)$$

holds, where the constant c does not depend on u, δ, and

$$q^* = \max_{k\in K}\sum_{p=1}^{s}(k_p - [k_p/2]).$$

Theorem 3. *Let $0 \leqslant s \leqslant n$, and let the differential operator P satisfy the conditions of Theorem 2. Then all generalized solutions of equation (1) on the domain $\Omega_s \times E_{n-s}$ satisfying condition (3) are infinitely differentiable functions after modification on a set of measure zero, and inequality (5) holds.*

For $s = 0$ and $s = 1$ the differential operators in Theorem 3 may be non-hypoelliptic. The simplest example of a non-hypoelliptic equation satisfying the conditions of Theorem 3, such that all its generalized solutions satisfying condition (3) are infinitely differentiable, is (for $n = 2,\ s = 1$) the equation $\partial^4 u/\partial x_2^4 + \partial^4 u/\partial x_1^2\partial x_2^2 - \partial^2 u/\partial x_1^2 = 0$. The first examples of this kind of equations were given by Ya. S. Bugrov ([5]). In the author' s paper ([6]) a special case of Theorem 3 ($n = 2,\ s = 1$) was proved under additional assumptions: all points of the set K have even coordinates, $\operatorname{sign} a_k = \operatorname{sign}(-1)^{[k/2]}$. It was also shown there that, under these assumptions, the conditions of Theorem 3 are equivalent to the conditions

$$\frac{\partial\mathcal{P}(\xi_1,\xi_2)}{\partial\xi_1}\Big/\mathcal{P}(\xi_1,\xi_2)\underset{\xi_1^2+\xi_2^2\to\infty}{\longrightarrow} 0, \qquad \left|\frac{\partial\mathcal{P}(\xi_1,\xi_2)}{\partial\xi_2}\Big/\mathcal{P}(\xi_1,\xi_2)\right|_{\xi_1^2+\xi_2^2\to\infty} < c_1(\xi_1^2+\xi_2^2)^{c_2},$$

where

$$\mathcal{P}(\xi) = \sum_{k \in K} a_k (i\xi)^k;$$

c_1 and c_2 are some positive constants.

For the proof of Theorem 3 (when $1 \leqslant s \leqslant n$) inequality (4) is sufficient. Inequality (6), refining (in the case of nondegenerate operators) inequality (4), makes it possible to obtain an estimate of the constant $c_\delta^{(l)}$ in inequality (5) as a function of l and δ, and to study the analyticity of solutions of equation (1) satisfying condition (3).

Theorem 4. *Let $s = 0, 1,$ or n; let P be a nondegenerate differential operator of order m; when $s = 0$, condition 1) of Theorem 1 is fulfilled; when $s = 1$, in addition, for $p = 2, \dots, n$, $\max_{k \in K} k_p = m$; when $s = n$, $\max_{k \in K} k_p = m$ for every $p = 1, \dots, n$.*

Then all generalized solutions of equation (1) on $\Omega_s \times E_{n-s}$ satisfying condition (3) are, after modification on a set of measure zero, analytic functions.

For $s = n$, Theorem 4 gives a new proof of I. G. Petrovskii' s theorem on the analyticity of solutions of elliptic equations. For $s = 0$ and $s = 1$, the differential operators in Theorem 4 may be nonelliptic (and even nonhypoelliptic). The simplest example of a nonelliptic equation satisfying the conditions of Theorem 4, for which all generalized solutions satisfying condition (3) are analytic, is the equation $\partial^4 u / \partial x_2^4 - \partial^2 u / \partial x_1^2 = 0$ $(n = 2,\ s = 1)$. A result analogous to Theorem 4 is also valid for the heat equation.

Theorem 5. For generalized solutions of the equation

$$\frac{\partial u}{\partial t} = \sum_{p=1}^{n} \frac{\partial^2 u}{\partial x_p^2},$$

considered in the strip

$$R_{(0,T)} = \{-\infty < x_p < \infty, \quad p = 1, \dots, n, \quad 0 < t < T\},$$

satisfying the condition

$$u \in L_2[R(T', T'')] \quad \text{for any } 0 < T' < T'' < T, \tag{7}$$

the inequalities

$$\left\| \frac{\partial u}{\partial x_p} \right\|_{L_2[R(T'+\delta, T''-\delta)]} \leq \frac{1}{\sqrt{2\delta}} \|u\|_{L_2[R(T', T'')]},$$

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_2[R(T'+\delta, T''-\delta)]} \leq \frac{1}{\delta} \|u\|_{L_2[R(T', T'')]}.$$

hold.

All generalized solutions of the equation

$$\frac{\partial u}{\partial t} = \sum_{p=1}^{n} \frac{\partial^2 u}{\partial x_p^2},$$

satisfying condition (7), are, after modification on a set of measure zero, analytic functions.

The results formulated above can also be extended to the case of equations with variable coefficients.

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Note: Figure translations are in progress. See original paper for figures.