THE OCCURRENCE OF DISCONTINUITIES IN THE CONTINUATION OF SOLUTIONS OF NONLINEAR MIXED PROBLEMS FOR THE STRING EQUATION

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V. N. GOL' DBERG

THE OCCURRENCE OF DISCONTINUITIES IN THE CONTINUATION OF SOLUTIONS OF NONLINEAR MIXED PROBLEMS FOR THE STRING EQUATION

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1°. In the present note the following nonlinear mixed problem is considered:

$$u_{xx} - u_{tt} = -\Phi(x,t,u); \tag{1}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \quad \text{for } 0 \le x \le 1; \tag{2}$$

$$a_0(u)u_x + b_0(u)u_t = f_0(t,u) \quad \text{for } x = 0; \tag{3}$$

$$a_1(u)u_x + b_1(u)u_t = f_1(t,u) \quad \text{for } x = 1. \tag{4}$$

In ([1],[2]) it was shown that if the function $u \in C_2^0(\overline{\Pi}_{T_0})$ is a solution of problem (1)–(4) in the rectangle

$$\Pi_{T_0} = \{0 \le x \le 1,\ 0 \le t \le T_0\} \quad (0 < T_0 < +\infty)$$

and

$$h_i(u(i,T_0)) \equiv b_i(u(i,T_0)) - (-1)^i a_i(u(i,T_0)) \ne 0 \quad (i = 0,1), \tag{5}$$

then the solution u can be continued uniquely in the rectangle $\Pi_{T_0+\Delta T}$ $(\Delta T > 0)$ with preservation of smoothness. Inequality (5) is an unnatural restriction in a number of physical problems (see, for example, ([3])). Below the solvability of problem (1)–(4) is investigated in the case when, upon continuation in t of a smooth solution, inequality (5) for $i = 0$ becomes an equality at some moment $t = T^* \ne +\infty$. It turns out that for $t > T^*$ the solution of problem (1)–

(4) becomes nonunique, and discontinuities and loss of smoothness of solutions arise; moreover, increasing the smoothness and compatibility of the data of the problem (the functions $\Phi, \varphi, \psi, a_i, b_i, f_i$) does not lead to unique continuability of the solution and to preservation of smoothness for $t > T^*$. In this connection, in the present work discontinuous and piecewise-smooth solutions of problem (1)–(4) are constructed for $t > T^*$. For the examples given in §4 it is shown that the discontinuous and piecewise-smooth solutions introduced below arise naturally if one studies the behavior of solutions of mixed problems obtained under small perturbations of the operator $a_0(u)u_x + b_0(u)u_t$ and of the initial conditions, as the perturbations tend to zero.

Choose an arbitrary $0 < T < 1$. Put

$$R^1 = (-\infty, +\infty), \qquad D_T^0 = \overline{\Pi}_T \times R^1, \qquad D_T^1 = [0, T] \times R^1.$$

Let:

1) the functions $\Phi \in C_1(D_1^0)$, $f_i \in C_1(D_1^1)$, $a_i, b_i \in C_1(R^1)$, $\varphi \in C_2[0,1]$, $\psi \in C_1[0,1]$;
2) the function h_0 have only isolated and simple zeros on the interval R^1, while the function h_1 have no zeros on the interval R^1.

Suppose that in the rectangle

$$\Pi_{T^*} = \{0 \le x \le 1,\ 0 \le t < T^*\},$$

where $0 < T^* < 1$,* there exists a solution $u \in C_2^0(\Pi_{T^*})$ of problem (1)–(4), and moreover

$$\sup_{\Pi_{T^*}} |u| < +\infty, \qquad h_0(u(0,t)) \ne 0 \quad \text{for } 0 \le t < T^*, \qquad h_0(u(0,t)) \to 0$$

as $t \to T^*$.

* Obviously, the inequality $0 < T^* < 1$ does not restrict the generality.

Lemma 1. There exists $\lim_{t \to T^*} \overset{0}{u}(0,t) = u^*$, and $h_0(u^*) = 0$.

2°. In the domain $D_{T^*}^1$ define the function

$$P(t,u) = f_0(t,u) - a_0(u)[\varphi'(t) + \psi(t)] - a_0(u) \int_0^t \Phi(t-\tau, \tau, u(t-\tau,\tau))\, d\tau.$$

It can be shown that $P \in C_1(D_{T^*}^1)$. Put

$$P_0 = P(T^*, u^*), \quad P_{1,0} = P_t'(T^*, u^*), \quad P_{0,1} = P_u'(T^*, u^*),$$

$$\overset{0}{Q}_\delta = \{0 \le x \le 1,\ 0 \le t \le \min(x + T^* - \delta, T^*)\}, \quad 0 < \delta \le T^*.$$

Note that $|P_0| < +\infty$.

Theorem 1. Whatever P_0 may be, $\overset{0}{u} \in C(\overline{\Pi}_{T^*})$.*

Theorem 2. Let $P_0 \neq 0$. Then $\overset{0}{u} \in C_2(\overline{Q}_\delta)$ for $\delta > 0$,

$$\sup_{\overline{Q}_\delta} |D^1 u| \to \infty$$

as $\delta \to 0$, and

$$\sup_{\Pi_{T^*}} \left|D^1 u(x,t)\right| \sqrt{|t - (x + T^*)|} < +\infty.$$

One can construct examples of nonlinear mixed problems with arbitrarily smooth and well-compatible data for which Theorem 2 holds.

Theorem 3. Let $P_0 = 0,\ P_{1,0} \neq 0$. Then $\overset{0}{u} \in C_2(\overline{Q}_\delta)$ for $\delta > 0$ and $\overset{0}{u} \in C_1(\overline{\Pi}_{T^*})$.

Choose an arbitrary $T^* < T < 1$. Put

$$\Omega_T = \{(x,t) \in \Pi_T,\ t \neq x + T^*\},$$

$$G_T^0 = \{0 \leq x < T - T^*,\ x + T^* < t \leq T\},$$
$$G_T^1 = \Omega_T \setminus G_T^0.$$

Let $\mathfrak{M}_T$ be the set of functions $u \in C(\overline{\Pi}_T)$ such that:

1) $u \in C_1(\overline{G}_T^j)$ $(j = 0,1)$, $u_x + u_t \in C(\overline{\Pi}_T)$, and, whatever closed domain $\overline{D} \subseteq \Omega_T$ may be, the function $u \in C_2(\overline{D})$;

2) $u \equiv \overset{0}{u}$ in the rectangle $\overline{\Pi}_{T^*}$.

A function $u \in \mathfrak{M}_T$ will be called a solution of problem (1)–(4) in the rectangle $\overline{\Pi}_T$ if it satisfies equation (1) in the domain Ω_T and equations (3), (4) for $0 \leq t \leq T$.**

Theorem 4. Let $P_0 = 0,\ P_{1,0} \neq 0$, and $P_{1,0} h_0'(u^*) > 0$. Then there exists a $T^* < T < 1$ such that in the set $\mathfrak{M}_T$ there exist two and only two solutions u_i $(i = 1,2)$ of problem (1)–(4) in the rectangle $\overline{\Pi}_T$, moreover

$$u_1 \in C_1(\overline{\Pi}_T), \qquad u_2 \in C_1(\Pi_T).$$

Theorem 5. Let $P_0 = 0,\ P_{1,0} \neq 0,\ P_{1,0} h_0'(u^*) < 0$*** and

$$P_{0,1}^2 + 4 P_{1,0} h_0'(u^*) \neq 0.$$

Then there exists a $T^* < T < 1$ such that in the set $\mathfrak{M}_T$ there exists an infinite set of solutions of problem (1)–(4).

One can construct examples of nonlinear mixed problems with arbitrarily smooth and well-compatible data for which Theorems 4 and 5 hold.

3°. Below, discontinuous solutions of problem (1)–(4) are constructed for $t > T^*$ for any P_0. The introduction of discontinuous solutions in the case $P_0 \neq 0$ is caused by the absence of continuous solutions in the rectangle $\overline{\Pi}_T$ for any $T > T^*$. Indeed, independently of the value of P_0, for any T $(0 < T \leq 1)$ one can define the notion of a continuous generalized solution (c.g.s.) of problem (1)–(4) in the rectangle $\overline{\Pi}_T$. The function $\overset{0}{u}$ is the unique c.g.s. in the rectangle $\overline{\Pi}_T$ for any $0 < T \leq T^*$, and in the case $P_0 = 0$, for the corresponding T, all solutions are such,

* That is, the function u admits a continuous prolongation as $t \to T^*$.

** The derivatives $D^1 u$ at the point $(0, T^*)$ should be understood as the corresponding one-sided derivatives.

*** It can be shown that if $P_0 = 0$, $P_{1,0} \neq 0$, and $P_{1,0} h_0'(u^*) < 0$, then $P_{0,1} \neq 0$.

specified in Theorems 4 and 5. It can be shown that in the case $P_0 \neq 0$ problem (1)–(4) does not have a n.g.s. in the rectangle $\overline{\Pi}_T$ for any $T > T^*$, and increasing the smoothness and compatibility of the data of the problem does not lead to the existence of a n.g.s. The advisability of introducing discontinuous solutions in the case $P_0 = 0$ is indicated by Example 2 given in §4.

Put

$$H_0(u) = \int_{\varphi(0)}^{u} h_0(\xi)\, d\xi, \qquad J(t; u) = \int_0^t P(\tau, u(0, \tau))\, d\tau.$$

Lemma 2. Whatever P_0 may be,

$$H_0(u(0, t)) = J(t; u) \qquad \text{for } 0 \leq t \leq T^*.$$

Suppose there exists at least one value $\overline{u}^*$ different from u^* such that

$$H_0(\overline{u}^*) = J(T^*; u), \qquad H_0'(\overline{u}^*) = h_0(\overline{u}^*) > 0.$$

Fix an arbitrary such value $\overline{u}^*$. Choose any $T^* < T < 1$.

Denote by $\mathfrak{R}_T(\overline{u}^*)$ the set of functions such that: 1) $u \in C_2(\overline{G_T^0})$, $u \in C(\overline{G_T^1})$, $u(0, T^* + 0) = \overline{u}^*$; 2) $u \equiv \overset{0}{u}$ in the rectangle $\overline{\Pi}_{T^*}$.

We note that a function $u \in \mathfrak{R}_T(\overline{u}^*)$ has a discontinuity of the first kind on the line $t = x + T^*$ in a neighborhood of the point $(0, T^*)$.

Theorem 6. Let $P_0 = 0$. Then there exists $T^* < T < 1$ such that in the set $\mathfrak{R}_T(\overline{u}^*)$ there exists a unique function $u \in C_1(\overline{G_T^1})$ which satisfies equation (1) in the domain Ω_T and equations (3), (4) for $0 \le t \le T$.

Theorem 7. Let $P_0 \ne 0$. Then there exists $T^* < T < 1$ such that in the set $\mathfrak{R}_T(\overline{u}^*)$ there exists a unique function u having the following properties:

1) $|D^1 u(x,t)| \to \infty$ as $t - (x + T^*) \to -0$,

$$\sup_{G_T^1} |D^1 u(x,t)| \sqrt{|t - (x + T^*)|} < +\infty.$$

2) the function u satisfies equation (1) in the domain Ω_T and equations (3), (4) for $t \in [0, T^*)$, $t \in [T^*, T]$, and $t \in [0, T]$, respectively.

§4. In the examples given below the following nonlinear mixed problem is considered

$$u_{xx} = u_{tt}; \tag{6}$$

$$u(x,0) = 0, \qquad u_t(x,0) = \psi(x) \qquad \text{for } 0 \le x \le 1; \tag{7}$$

$$-u_x + \frac{\partial f(u)}{\partial u} u_t = 0 \qquad \text{for } x = 0; \tag{8}$$

$$u_x = 0 \qquad \text{for } x = 1, \tag{9}$$

where $f(u) = \frac{5}{3}u^3 - \frac{5}{2}u^2 + \frac{11}{6}$, and $\psi(x)$ is a function of class $C_1[0,1]$, chosen in each example in a special way and satisfying the compatibility conditions $\psi^{(s)}(j) = 0$ $(s, j = 0, 1)$. Note that $h_1(u) \equiv 1$, while the equation $h_0(u) = 0$ has two real roots

$$u_j^* = \frac{1}{2} + (-1)^j \frac{\sqrt{5}}{10} \qquad (j = 1, 2).$$

Example 1. Choose an arbitrary function ψ such that: a) $\psi(x) > 0$ for $0 < x \le \frac{3}{4}$; b) the function

$$\sigma(x) = f(0) + \int_0^x \psi(\xi)\, d\xi$$

is such that

$$\sigma(x) < u_1^* + f(u_1^*) \quad \text{for } 0 < x < \frac{1}{2}$$

and

$$\sigma\left(\frac{1}{2}\right) = u_1^* + f(u_1^*).$$

Theorem 8. Problem (6)–(9) has in the rectangle $\Pi_{1/2}$ a unique solution $u \in C_2(\Pi_{1/2})$, moreover

$$h_0(u(0,t)) > 0 \quad \text{for } 0 \leq t < \frac{1}{2},$$

$$u(0,t) \to u_1^* \quad \text{as } t \to \frac{1}{2}, \qquad P_0 \neq 0.$$

In the example under consideration $T^* = \frac{1}{2}$. It can be shown that the equation

$$H_0(u) = J(T^*; u)$$

has a unique root $\overline{u}^*$, different from the root u_1^*. Consequently, by Theorem 7, there exists $T^* < T < 1$ such that the problem

(6)–(9) has in the rectangle Π_T a unique discontinuous solution $u_p \in \mathfrak{R}_T(\bar{u}^*)$. Let us now consider equation (6) with the initial conditions (7), the boundary condition (9), and the following boundary condition:

$$\mu u_{tt} - u_x + \frac{\partial f(u)}{\partial u}\, u_t = 0 \quad (\mu > 0) \quad \text{for } x = 0. \tag{10}$$

Theorem 9. There exists a $\mu_0 > 0$ such that, for any $0 < \mu < \mu_0$, problem (6), (7), (9), (10) has in the rectangle $\overline{\Pi}_T$ a unique solution $u_\mu \in C_2(\overline{\Pi}_T)$, and $u_\mu \to u_p$ as $\mu \to 0$ uniformly with respect to x, t in each closed domain $\bar{D} \subset \Omega_T$.

Example 2. Choose an arbitrary function ψ satisfying condition b) and such that $\psi(x) > 0$ for $0 \leq x < 1/2$, $\psi(1/2) = 0$, $\psi'(1/2) < 0$.

Theorem 10. In the rectangle $\Pi_{1/2}$ there exists a unique solution $u \in C_2(\Pi_{1/2})$ of problem (6)–(9), and $h_0(u(0,t)) > 0$ for $0 \leq t < 1/2$, $u(0,t) \to u_1^*$ as $t \to 1/2$; $P_0 = 0$, $P_{1,0} \neq 0$, $P_{1,0} h_0'(u_1^*) > 0$.

As in Example 1, $T^* = 1/2$.

Theorem 11. Let T be the number indicated in Theorem 4. For any $\varepsilon > 0$, one can specify a $\mu_0(\varepsilon) > 0$ such that, for any $0 < \mu < \mu_0(\varepsilon)$, problem (6), (7), (9), (10) has in the rectangle $\overline{\Pi}_T$ a unique solution $u_\mu \in C_2(\overline{\Pi}_T)$, and

$$\max_{\overline{\Pi} T} |u_\mu - u_2| < \varepsilon,$$

where u_2 is the solution of problem (6)–(9) appearing in Theorem 4.

As in Example 1, the existence is established of such a $T^* < \bar{T} < 1$ that in the rectangle $\overline{\Pi}_{\bar{T}}$ problem (6)–(9) has a unique discontinuous solution $u_p \in \mathfrak{R}_{\bar{T}}(\bar{u}^*)$, where $\bar{u}^*$ is the unique root, distinct from u_1^*, of the equation $H_0(u) = J(T^*; u)$.

Theorem 12. Whatever closed domain $\bar{D} \subset G_{\bar{T}}^1$ is given, for any $\varepsilon > 0$ one can specify a $\delta_0(\varepsilon) > 0$ such that, for any $0 < \delta < \delta_0(\varepsilon)$, there exist $0 < T^*(\delta) < T^*$ and a function $\psi_\delta(x) \in C_1[0, 1]$ satisfying the relations

$$\bar{D} \subset \Gamma_\delta = \{0 \leq x \leq 1,\ 0 \leq t \leq \min(x + T^*(\delta), \bar{T})\},$$

$$\sum_{s=0,1} \max_{0 \leq x \leq 1} |\psi_\delta^{(s)}(x) - \psi^{(s)}(x)| < \delta$$

and such that problem (6)–(9), with initial conditions $u(x, 0) = 0$, $u_t(x, 0) = \psi_\delta(x)$:

1) has in the rectangle $\overline{\Pi}_{T^*(\delta)}$ a unique solution $u^\delta \in C_2(\overline{\Pi}_{T^*(\delta)})$, with $h_0(u^\delta(0, t)) > 0$ for $0 \leq t < T^*(\delta)$, $u^\delta(0, t) \to u_1^*$ as $t \to T^*(\delta)$; $P_0 \neq 0$;
2) has in the rectangle $\overline{\Pi}_{\bar{T}}$ a unique discontinuous solution $u_p^\delta \in \mathfrak{R}_{\bar{T}}(\bar{u}_\delta^*)$*, where $\bar{u}_\delta^*$ is the unique root, distinct from u_1^*, of the equation $H_0(u) = J(T^*(\delta); u^\delta)$, and

$$\max_{\Gamma_\delta} |u_R - u_p^\delta| < \varepsilon, \qquad \max_{\substack{0 \leq x \leq \bar{T} - T^* \\ x + T^* \leq t \leq \bar{T}}} |u_p - u_p^\delta| < \varepsilon.$$

Scientific Research Radiophysics Institute
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* Here, in the definition of the domains Ω_T, G_T^i $(i = 0, 1)$ and the set $\mathfrak{R}_{\bar{T}}(\bar{u}_\delta^*)$, one should put $T^* = T^*(\delta)$, $\bar{u}^* = \bar{u}_\delta^*$, $u = u^\delta$.

Note: Figure translations are in progress. See original paper for figures.