MATRIX ELEMENTS OF THE LORENTZ TRANSFORMATION FOR A UNITARY REPRESENTATION

DAO VONG DYK, NGUYEN VAN HIEU

1967

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICAL PHYSICS

DAO VONG DYK, NGUYEN VAN HIEU

MATRIX ELEMENTS OF THE LORENTZ TRANSFORMATION FOR A UNITARY REPRESENTATION

(Presented by Academician N. N. Bogolyubov on 18 VI 1966)

At the present time the possibility of applying a noncompact symmetry group to the classification of elementary particles is being intensively discussed $^{(1-5)}$. It has been pointed out that this group must contain as a subgroup $SL(2,C)$, isomorphic to the Lorentz group. In connection with this there is increasing interest in the study of the Lorentz group, especially its unitary (infinite-dimensional) representations.

In the present work we shall find an expression for the matrix element of proper Lorentz transformations in the case of a unitary representation. These matrix elements must be known in studying the structure of the matrix elements of processes in any symmetry with noncompact groups $^{(5)}$. We note that this problem was first posed and partially solved (for the case $j_0 = 0$) by A. Z. Dolginov and I. N. Toptygin in $^{(6)}$. The authors of that work used, as basis functions, functions that are the analytic continuation of the four-dimensional spherical functions of Euclidean space (for more detail see $^{(6,7)}$).

Our derivation is based exclusively on the results of I. M. Gel' fand and M. A. Naimark $^{(8,9)}$. As is known $^{(9)}$, an irreducible representation of the proper Lorentz group may be characterized by a pair of numbers (j_0, c), where j_0 is an integral or half-integral nonnegative number representing the smallest weight of representations of the subgroup of three-dimensional rotations, and c is a complex number. We shall consider here the unitary irreducible representation of the principal series ($\sigma_{m,\rho}$ in the notation of $^{(9)}$; $j_0 = |m/2| \equiv |\nu_0|$, $c = -i(\operatorname{sign} m)\rho/2$ for $m \neq 0$; $j_0 = 0$, $c = \pm i\rho/2$ for $m = 0$), to which a purely imaginary c corresponds.

The canonical basis of the representation $\sigma_{m,\rho}$, realized in the Hilbert space $L_2(Z)$, is (for details see $^{(9)}$)

$$f_{j\mu}^{\nu_0\rho}(z) = \frac{1}{\sqrt{\pi}}\alpha^{-1}(u)\varphi_{j\mu}^{\nu_0\rho}(u), \tag{1}$$

where

$$\alpha^{-1}(u) = |u_{22}|^{m-i\rho+2} u_{22}^{-m},$$

$$\varphi_{j\mu}^{\nu_0\rho}(u) = \sqrt{2j+1}\,\chi_j^{\nu_0\rho}(-1)^{-2j-\nu_0-\mu}\frac{\sqrt{(j-\nu_0)!(j+\nu_0)!}}{\sqrt{(j-\mu)!(j+\mu)!}}\times$$

$$\times\sum_{d=\max(0,-\nu_0,-\mu)}^{\min(j-\nu_0,j-\mu)} C_{j-\mu}^{d} C_{j+\mu}^{j-\nu_0-d} u_{11}^{d} u_{12}^{j-\nu_0-d} u_{21}^{j-\mu-d} u_{22}^{\nu_0+\mu+d},$$

$$\chi_j^{\nu_0\rho} = \prod_{\nu=j_0}^{j}\frac{-2\nu+i\rho}{\sqrt{4\nu^2+\rho^2}};$$

C_p^q is the number of combinations of p elements taken q at a time; u, z belong to the so-called class $\tilde{z}$. In the space $L_2(z)$ the scalar product is given by the formula

$$\langle f_1|f_2\rangle = \int f_1(z)\overline{f_2(z)}\,dz, \qquad z = x+iy, \qquad dz = dx\,dy, \tag{2}$$

and to each Lorentz transformation that is described by a matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of the unimodular group of the second order there corresponds an operator U_a, given by the formula

$$U_a f(z) = |a_{12}z + a_{22}|^{-m+i\rho-2}(a_{12}z+a_{22})^m f\left(\frac{a_{11}z+a_{21}}{a_{12}z+a_{22}}\right). \tag{3}$$

Suppose we have to find the matrix element $D_{j\mu;\,j'\mu'}^{m\rho}(a)$, which is defined as

$$U_a|m\rho;\,j\mu\rangle = \sum_{j'\mu'} D_{j\mu;\,j'\mu'}^{m\rho}(a)|m\rho;\,j'\mu'\rangle, \tag{4}$$

where $|m\rho;\,j\mu\rangle$ denotes the canonical basis of the representation $\sigma_{m,\rho}$. From (4) and from the orthonormality condition

$$\langle f_{j\mu}^{\nu_0\rho} \mid f_{j'\mu'}^{\nu_0\rho}\rangle = \int f_{j\mu}^{\nu_0\rho}(z)\,\overline{f_{j'\mu'}^{\nu_0\rho}(z)}\,dz = \delta_{jj'}\delta_{\mu\mu'}$$

it follows that

$$D^{m\rho}_{j\mu;\,j'\mu'}(a) = \int U_a f^{\nu_0\rho}_{j\mu}(z)\,\overline{f^{\nu_0\rho}_{j'\mu'}(z)}\,dz. \tag{5}$$

It is known that every matrix a can be represented in the form

$$a = u_1 \varepsilon u_2,$$

where u_1, u_2 are unitary unimodular matrices corresponding to a three-dimensional rotation;

$$\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad (\varepsilon \text{ is a real number})$$

and corresponds to a pure Lorentz rotation in the plane (x_3, x_4). Thus, without loss of generality, one may restrict oneself to finding $D^{m\rho}_{j\mu;\,j'\mu'}(\varepsilon)$.

Substituting (1) and (3), with the values

$$u_{11} = \bar{u}_{22} = (1+|z|^2)^{-1/2} e^{-i\omega}, \qquad u_{12} = -\bar{u}_{21} = -\bar{z}(1+|z|^2)^{-1/2} e^{i\omega}$$

($e^{i\omega}$ is a certain phase factor), into (5), we obtain

$$\begin{aligned} D^{m\rho}_{j\mu;\,j'\mu'}(\varepsilon) = \frac{1}{\pi}\,\delta_{\mu\mu'} &\left\{(2j+1)(2j'+1)\frac{(j-\nu_0)!(j+\nu_0)!(j'-\nu_0)!(j'+\nu_0)!}{(j-\mu)!(j+\mu)!(j'-\mu)!(j'+\mu)!}\right\}^{1/2} \\ &\times \chi^{\nu_0\rho}_j \overline{\chi^{\nu_0\rho}_{j'}} \sum_{d,d'=\max(0,-\nu_0-\mu)}^{\min(j-\nu_0,j-\mu),\,\min(j'-\nu_0,j'-\mu)} (-1)^{d+d'} \\ &\times C^{d}_{j-\mu} C^{j-\nu_0-d}_{j+\mu} C^{d'}_{j'-\mu} C^{j'-\nu_0-d'}_{j'+\mu}\, \varepsilon^{4(j-d-\mu/2-\nu_0/2+1/2-i\rho/4)} \\ &\times \int dz\,|z|^{2(j+j'-d-d'-\mu-\nu_0)}(1+|z|^2)^{-i\rho/2-j'-1}(1+\varepsilon^4|z|^2)^{i\rho/2-j-1}. \end{aligned} \tag{6}$$

Passing to the polar coordinate system

$$x = r\cos\varphi, \qquad y = r\sin\varphi \qquad (0 \le r < \infty,\ 0 \le \varphi \le 2\pi)$$

and then using the substitution $v = r^2$, we reduce the integral in (6) to the form

$$I = \pi \int_0^\infty dv\, v^{j+j'-d-d'-\mu-\nu_0}(1+v)^{-i\rho/2-j'-1}(1+\varepsilon^4 v)^{i\rho/2-j-1}.$$

For the values of d, d' indicated in (6), I turns out to be equal to

$$I = \pi \, \frac{(j+j'-d-d'-\mu-\nu_0)!(d+d'+\mu+\nu_0)!}{(j+j'+1)!} \, \varepsilon^{4(d+d'+\mu+\nu_0-j+i\rho/2)} \\ \times F\left(j'+1+i\rho/2,\, d+d'+\mu+\nu_0+1;\, j+j'+2;\, 1-\varepsilon^4\right), \tag{7}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function.

Substituting (7) into (6), we obtain the final result:

$$\begin{aligned} D^{m\rho}_{j\mu;\, j'\mu'}(\varepsilon) = {} & \frac{\delta_{\mu\mu'}}{(j+j'+1)} \{(2j+1)(2j'+1)(j-\nu_0)!(j+\nu_0)! \times \\ & \times (j-\mu)!(j+\mu)!(j'-\nu_0)!(j'+\nu_0)!(j'-\mu)!(j'+\mu)!\}^{1/2} \times \\ & \times \chi_j^{\nu_0\rho} \, \overline{\chi_{j'}^{\nu_0\rho}} \sum_{d,d'} (-1)^{d+d'} \left[(j+j'-d-d'-\mu-\nu_0)!(d+d'+\mu-\nu_0)!\right] \times \\ & \times \big[d!\, d'!(j-\mu-d)!(j'-\mu-d')!(j-\nu_0-d)!(j'-\nu_0-d')! \times \\ & \times (\mu+\nu_0+d)!(\mu+\nu_0+d')!\big]^{-1} \\ & \times \varepsilon^{2(3d'+\mu+\nu_0+1+i\rho/2)} F(j'+1+i\rho/2,\ d+d'+\mu+\nu_0+1;\ j+j'+2;\ 1-\varepsilon^4), \end{aligned} \tag{8}$$

where d, d' run through the integers that do not make any factor under the factorial sign negative.

We now note some simple properties of the functions $D(\varepsilon)$.

1.

$$D^{m\rho}_{j\mu;\, j'\mu'}(1) = \delta_{jj'} \cdot \delta_{\mu\mu'}. \tag{9}$$

This equality follows directly from (8), taking into account that $F(\alpha, \beta; \gamma; 0) = 1$. It expresses the fact that we are dealing with the identity transformation.

2.

$$D^{m\rho}_{j'\mu';\, j\mu}(\varepsilon) = \overline{D^{m\rho}_{j\mu;\, j'\mu'}(\varepsilon^{-1})}. \tag{10}$$

This relation follows directly from the property of the hypergeometric function

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F(\gamma-\alpha, \beta; \gamma; z/(z-1)).$$

3. From (9), (10), and the group property of the functions D, it follows at once that

$$\sum_{j\mu} D^{m\rho}_{j\mu;\,j'\mu'}(\varepsilon)\,\overline{D^{m\rho}_{j\mu;\,j''\mu''}(\varepsilon)} = \delta_{j'j''}\delta_{\mu'\mu''}.$$

The last relation is the unitarity condition for the representation.

In conclusion, the authors express their deep gratitude to Academician N. N. Bogoliubov for his interest in the work.

United Institute
for Nuclear Research

Received
21 V 1966

References

[1] A. O. Barut, P. Budini, C. Fronsdal, Preprint, 1965.
[2] Y. Dothan, M. Gell-Mann, Y. Ne' eman. Phys. Lett., **17**, 148 (1965).
[3] C. Fronsdal, Preprint, 1965.
[4] R. Delbourgo, A. Salam, J. Strathdee, Proc. Roy. Soc., **289**, 177 (1965).
[5] H. Ruegg, in *Lectures at the International School on Theoretical Physics*, Yalta, 1966.
[6] A. Z. Dolginov, N. Toptygin, ZhETF, **37**, 1441 (1959).
[7] A. Z. Dolginov, ZhETF, **30**, 746 (1956).
[8] I. M. Gelfand, M. A. Naimark, Tr. Mat. Inst. im. V. A. Steklova, Academy of Sciences of the USSR, **36** (1950).
[9] M. A. Naimark, *Linear Representations of the Lorentz Group*, 1958.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.