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ON A PROPERTY OF PROPOSITIONAL FORMULAS

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Abstract

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MATHEMATICS

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ON A PROPERTY OF PROPOSITIONAL FORMULAS

(Presented by Academician P. S. Novikov on 6 VII 1966)

1. A. A. Markov stated the following proposition: whatever the propositional formulas P_1 and P_2 may be, if the formula $(P_1 \vee P_2)$ is realizable, then either P_1 is realizable or P_2 is realizable. In the present note it is shown that this proposition cannot be refuted by an example (Theorem 1 and Corollary 1). A similar assertion was proved classically by F. L. Varakhovskii⁽¹⁾, but the definition of a realizable formula in his work differs from the constructive variant of J. F. Rose's definition⁽²⁾ adopted here.
2. Let n be an arbitrary natural number. Fix some Gödel numbering of n -term systems of closed logical-arithmetical formulas. For convenience we shall assume that to each number there corresponds some system. By the symbol A_k^j ($1 \leq j \leq n$) we denote the j -th term of the system with Gödel number k . Let P be a propositional formula in the variables p_1, \dots, p_n , and let φ be a partial recursive function of one argument (p.r.f.-1). We shall say that φ realizes P if, for every k , $\varphi(k)$ is defined and realizes, in the sense of Kleene⁽³⁾, the formula $P(A_k^1, \dots, A_k^n)$ (here $P(A_k^1, \dots, A_k^n)$ denotes the result of substituting the formulas A_k^1, \dots, A_k^n into the formula P in place of the variables p_1, \dots, p_n , respectively). A propositional formula P will be called realizable if there exists a p.r.f.-1 φ realizing the formula P .
3. **Theorem 1.** *Whatever the p.r.f.-1 φ and the propositional formulas P_1 and P_2 may be, such p.r.f.-1 ψ_1 and ψ_2 can be constructed that, if φ realizes the formula $(P_1 \vee P_2)$, then there cannot fail to exist a number s such that: 1) $s = 1$ or $s = 2$; 2) ψ_s realizes P_s .*

We briefly outline the proof of Theorem 1. Construct a sequence M of recursively enumerable sets, taking $i \in M_k$ if and only if the formula

$$\exists y (T_1(i, i, y) \& U(y) = k)$$

is realizable (with respect to the predicates $T_1(a, x, y)$ and $U(y) = k$, see⁽⁴⁾, pp. 248 and 250). It can be shown that if $k \neq r$, then the sets M_k and M_r are disjoint and recursively inseparable⁽⁵⁾.

We denote by the symbol A_{im}^j the formula

$$\bigwedge_{k=1}^m (\exists y (T_1(i, i, y) \& U(y) = k) \supset A_k^j) \quad (1 \leq j \leq n).$$

It can be shown that if

$$i \in M_k, \quad m \geq k, \quad (1)$$

then the formulas $A_{im}^j \sim A_k^j$ are realizable and, consequently, for any propositional formula P in the variables p_1, \dots, p_n the formula

$$P(A_k^1, \dots, A_k^n) \sim P(A_{im}^1, \dots, A_{im}^n) \quad (2)$$

is realizable.

Let $\rho(i, m)$ be such a general recursive function that for any natural numbers i and m the number $\rho(i, m)$ is a Gödel number of the system $A_{im}^1, \dots, A_{im}^n$. Using the pairwise recursive inseparability of the sets of the sequence M , one can show that the following is true.

Lemma. Whatever the p.r.f.-1 φ and the propositional formulas P_1 and P_2 may be, there cannot fail to exist a number s such that: 1) $s = 0$ or $s = 1$; 2) if φ realizes the formula $(P_1 \vee P_2)$, then for every k there are natural numbers i, m , and c such that $i \in M_k$, $m \geq k$, and

$$\varphi(\rho(i, m)) = 2^s \cdot 3^c.$$

Using this lemma and the realizability of formula (2) under condition (1), one can carry out a proof of Theorem 1 that is consistent with both the classical and the constructive understanding of mathematical judgments. From Theorem 1, with the aid of the principle of constructive choice ⁽⁶⁾, there is derived

Corollary 1. Whatever the p.r.f.-1 φ and the propositional formulas P_1 and P_2 may be, one can construct such a p.r.f.-1 ψ_1 that, if φ realizes the formula $(P_1 \vee P_2)$ and P_2 is not realizable, then ψ_1 realizes P_1 .

Within classical logic, from Theorem 1 one can obtain

Corollary 2. If the propositional formula $(P_1 \vee P_2)$ is realizable, then the formula P_1 is realizable or the formula P_2 is realizable.

4. The proof of Theorem 1 found by the author does not, however, give an algorithm which, for each realizable propositional formula of the form $(P_1 \vee P_2)$, would indicate that one of the formulas P_1, P_2 which is realizable. The question of the existence of such an algorithm remains open.

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