AN EXAMPLE OF A HARMONIC FUNCTION BOUNDED OUTSIDE A BODY OF REVOLUTION AND GROWING INSIDE IT

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AN EXAMPLE OF A HARMONIC FUNCTION BOUNDED OUTSIDE A BODY OF REVOLUTION AND GROWING INSIDE IT

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The present note is devoted to the following problem, discussed in papers ($^{1-6}$). Let D be some unbounded domain in three-dimensional space (x, x_1, x_2), and let $u(P)$ be a function harmonic in this domain and continuous, together with its gradient, up to the boundary S of the domain D. It is required to estimate the upper bound $\varphi_D(P)$ of functions $\varphi(P)$ such that from the conditions

$$|u(P)| < C \exp\{\varphi(P)\}, \qquad P \in D,$$

$$|u(P)| + |\operatorname{grad} u(P)| < C, \qquad P \in S,$$

there follows the boundedness of $u(P)$ everywhere inside D.

We shall call the function $\varphi_D(P)$ the **growth indicator** for the domain D.

In the paper of M. A. Evgrafov (^{6}), a lower estimate for the growth indicator is found in the case of an arbitrary domain whose boundary satisfies certain smoothness conditions. In the note (^{4}), an example is constructed of a harmonic function in the whole space, bounded outside a circular half-cylinder and unbounded inside. The growth of this function gives an upper estimate for the growth indicator for every domain containing within itself a circular half-cylinder. Naturally, this estimate gives a satisfactory result only for domains which themselves are close to a half-cylinder.

In the present note, by somewhat generalizing the construction proposed in (^{4}), we obtain an upper estimate for the growth indicator sufficiently sharp for a broader class of domains.

Theorem 1. Let the domain V be a body of revolution with generatrix

$$x_1 = \tfrac{1}{2}h(x); \qquad x > x_0; \qquad h(x) \geq r > 0.$$

Suppose that the function $h(x)$ satisfies the following conditions:

1. $h(z)$ is regular and has no zeros in the domain G_{x_0}:

$$|\operatorname{Im} z| < \tfrac{1}{2}h(\operatorname{Re} z), \qquad \operatorname{Re} z > x_0.$$

2. In the indicated domain,

$$\lim_{|z|\to\infty} h'(z) = 0.$$

Then there exists a function $u_\eta(x, x_1, x_2)$, harmonic in the whole space, bounded outside the body of revolution V, and unbounded inside, and moreover

$$|u_\eta(x, x_1, x_2)| < C \exp\{\varphi_{2\eta}(x)\}; \qquad (x, x_1, x_2) \in V,$$

where

$$\varphi_\eta(x) = \exp\left\{(\pi + \eta)\int_{x_0}^{x} \frac{dt}{h(t)}\right\},$$

and $\eta > 0$ is arbitrarily small.

Proof. Put

$$\varphi_\eta(z) = \exp\left\{(\pi + \eta)\int_{x_0}^{z} \frac{dt}{h(t)}\right\}.$$

We shall further denote by $G_{\alpha,x}$ the domain of the complex z-plane:

$$|\operatorname{Im} z| < \frac{\alpha}{2}h(\operatorname{Re} z), \qquad \operatorname{Re} z > x,$$

and by $L_{\alpha,x}$ the contour bounding this domain. Finally, let

$$u_\eta(x, x_1, x_2) = \frac{1}{2\pi i}\int_{L_{\alpha,x_0+1}} \frac{\exp\{\varphi_\eta(z)\}}{\sqrt{(z-x)^2+\rho^2}}\,dz; \qquad \rho = \sqrt{x_1^2 + x_2^2}, \quad x < x_0, \tag{1}$$

where the regular branch of the root is fixed by the condition

$$\operatorname{Re}\sqrt{(z-x)^2+\rho^2} > 0,$$

and $\alpha < 1$ is chosen so that the inequality

$$\pi > \frac{\pi+\eta}{2}\alpha > \frac{\pi}{2}$$

is satisfied.

From the conditions imposed on $h(z)$ there easily follows the estimate

$$\left|\exp\left\{\varphi_\eta\left(x \pm i\frac{\alpha}{2}h(x)\right)\right\}\right| = \exp\left\{\varphi_\eta(x)\cos\left[\frac{\pi+\eta}{2}\alpha(1+o(1))\right]\right\}, \qquad x \to \infty.$$

It is also clear that, as $x \to \infty$, the function $\varphi_\eta(x)$ grows faster than any power of x. Therefore the integral (1) converges absolutely. Differentiating under the integral sign, we are convinced that $u_\eta(x, x_1, x_2)$ is a harmonic function in the half-space $x < x_0$; replacing in (1) the contour of integration L_{α,x_0+1} by $L_{\alpha,x+1}$, we carry out a harmonic continuation of this function to the entire space (x, x_1, x_2).

Let us show that $u_\eta(x, x_1, x_2)$ is bounded outside the body V. Indeed, in this case either $x < x_0$, or $\rho \geq \frac{1}{2}h(x)$. Hence the branch points of the root $z = x \pm i\rho$ lie outside the domain G_{x_0}. But G_{x_0} contains within itself the contour L_{α,x_0+1}, since $\alpha < 1$. Therefore

$$u_\eta(x, x_1, x_2) = \frac{1}{2\pi i}\int_{L_{\alpha,x_0+1}} \frac{\exp\{\varphi_\eta(z)\}}{\sqrt{(z-x)^2+\rho^2}}\,dz; \quad \rho = \sqrt{x_1^2+x_2^2}; \quad (x, x_1, x_2) \notin V. \tag{2}$$

Since

$$\left|(z-x)^2+\rho^2\right| = |z-(x+i\rho)| \cdot |z-(x-i\rho)|,$$

where $z \in L_{\alpha,x_0+1}$, while $x \pm i\rho \notin G_{x_0}$, we have

$$\left|\sqrt{(z-x)^2+\rho^2}\right| \geq d > 0,$$

where d is the distance between L_{α,x_0+1} and L_{x_0}. According to (2), this gives

$$|u_\eta(x, x_1, x_2)| \leq \frac{1}{2\pi d}\int_{L_{\alpha,x_0+1}} |\exp\{\varphi_\eta(z)\}|\,|dz| \leq \text{const}.$$

Thus, the boundedness of $u_\eta(x, x_1, x_2)$ outside V is proved.

Now let us estimate from above the growth of $u_\eta(x, x_1, x_2)$ inside the body V. Obviously,

$$|u_\eta(x, x_1, x_2)| \leq \left| \int_{x+1-\frac{i\alpha}{2}h(x+1)}^{x+1+\frac{i\alpha}{2}h(x+1)} \exp\{\varphi_\eta(z)\}\, dz \right| + \text{const} \leq$$

$$\leq Ch(x+1)\exp\{\varphi_\eta(x+1)\};$$

therefore

$$|u_\eta(x, x_1, x_2)| < C\exp\{\varphi_{2\eta}(x)\}.$$

It remains to establish that the function $u_\eta(x, x_1, x_2)$ constructed by us is not bounded inside V. Putting $\rho = 0$, we have

$$u_\eta(x, 0, 0) = \frac{1}{2\pi i}\int_{L_{\alpha, x+1}} \frac{\exp\{\varphi_\eta(z)\}}{z-x}\, dz = \exp\{\varphi_\eta(x)\} + \frac{1}{2\pi i}\int_{L_{\alpha, x_0+1}} \frac{\exp\{\varphi_\eta(z)\}}{z-x}\, dz.$$

The integral is bounded, while $\exp\{\varphi_n(x)\}$ grows as $x \to \infty$. The theorem is proved.

From the result obtained it follows that, for any domain D containing within itself the body of revolution V specified in the statement, we have

$$\varphi_D(P) \leqslant \exp\left\{(\pi+\varepsilon)\int_{x_0}^{x} \frac{dt}{h(t)}\right\}$$

with arbitrarily small $\varepsilon > 0$.

To judge the sharpness of the estimate obtained, we give one assertion which follows easily from the results of M. A. Evgrafov[6] and the known Phragmén–Lindelöf theorems for regular functions:

Theorem 2. *Suppose that the function $h(x)$, in addition to the conditions indicated above, also has the following property:*

$$\int_{x_0}^{\infty} \frac{{h'}^2(t)}{h(t)}\, dt < \infty,$$

and suppose that the domain D is entirely contained inside the cylinder with directrix $x_1 = \frac{1}{2}h(x)$ and generators parallel to the axis Ox_2 (in particular, our body of revolution V may serve as the domain D). Then

$$\varphi_D(P) \geqslant \exp\left\{(\pi-\varepsilon)\int_{x_0}^{x} \frac{dt}{h(t)}\right\}$$

with any $\varepsilon > 0$.

In conclusion, the authors of the present note express their gratitude to M. A. Evgrafov for his constant attention and interest in this work.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.