ONE GENERALIZATION OF THE POWER MOMENT PROBLEM

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.948.35

MATHEMATICS

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ONE GENERALIZATION OF THE POWER MOMENT PROBLEM

(Presented by Academician A. Yu. Ishlinskii on 26 III 1966)

This note investigates a certain operator moment problem arising from the desire to satisfy the axioms of field theory proposed by Wightman ((1); see also Borchers (2)). In the scheme considered here, so far only a small number of the requirements of these axioms are reflected.

1°. Let $\mathfrak{H}$ be a Hilbert space, $\mathfrak{H} \otimes \cdots \otimes \mathfrak{H} = \otimes \mathfrak{H}^j$ the tensor product of j copies of the space $\mathfrak{H}$ ($j = 0, 1, \ldots$; for $j = 0$, $\otimes \mathfrak{H}^0 = C_1$, where C_1 is the field of complex numbers). Vectors of the space $\otimes \mathfrak{H}^j$ will be denoted by $f_j, g_j, \ldots$; $\|f_j\|$, (f_j, g_j) are the norm and scalar product in $\otimes \mathfrak{H}^j$. In what follows all spaces are assumed separable.

Consider an operator K_{jk}, acting continuously from $\otimes \mathfrak{H}^k$ into $\otimes \mathfrak{H}^j$, and form the matrix $K = (K_{jk})_{j,k=0}^{\infty}$. Such a matrix will be called positive definite (p.d.) if, for any finite sequence $f = (f_0, f_1, \ldots)$, $f_j \in \otimes \mathfrak{H}^j$ ($\Phi(\mathfrak{H})$ denotes their aggregate),

$$\sum_{j,k=0}^{\infty} (K_{jk} f_k, f_j) \geq 0. \tag{1}$$

This definition is preserved if in each $\otimes \mathfrak{H}^k$ some subspace is distinguished and K_{jk} acts continuously from this subspace into the corresponding subspace of $\otimes \mathfrak{H}^j$ (then, clearly, the f_j in (1) must belong to the distinguished subspaces).

Introduce on sequences $f = (f_0, f_1, \ldots)$ the right shift

$$Tf = T(f_0, f_1, \ldots) = (0, f_0 \otimes e, f_1 \otimes e, \ldots), \tag{2}$$

where $e \in \mathfrak{H}$ is a fixed vector; in what follows, without loss of generality, one may assume that $\|e\| = 1$. A p.d. matrix will be called moment, if

$$\sum_{j,k=0}^{\infty} (K_{j+1,k} f_k, g_j \otimes e) = \sum_{j,k=0}^{\infty} (K_{j,k+1} f_k \otimes e, g_j) \quad (f, g \in \Phi(\mathfrak{H})), \tag{3}$$

i.e. the shift (2) on finite sequences is Hermitian in the scalar product

$$\langle f, g\rangle = \sum_{j,k=0}^{\infty} (K_{jk} f_k, g_j).$$

In the case of a one-dimensional space $\mathfrak{H}$, for any $j = 0, 1, \ldots$, $\otimes\mathfrak{H}^j = C_1$; $f = (f_0, f_1, \ldots)$ is a sequence of numbers; $(K_{jk})_{j,k=0}^{\infty}$ is an ordinary matrix. If it is moment for $e = 1$, then from (3) it follows that necessarily $K_{jk} = s_{j+k}$, where s_j is an ordinary moment sequence $(j, k = 0, 1, \ldots)$. As is known,

$$s_j = \int_{-\infty}^{\infty} \lambda^j \, d\tau(\lambda) \quad (j = 0, 1, \ldots),$$

where $d\tau(\lambda)$ is a certain nonnegative finite measure. We shall now show that an analogous representation also holds for a general moment matrix.

Denote by $\dot{\mathfrak{H}}$ the one-dimensional subspace of $\mathfrak{H}$ spanned by the vector e, and by $\overset{\infty}{\mathfrak{H}}$ its orthogonal complement. Let $A_{\alpha\beta}$ be an operator acting continuously from $\otimes\mathfrak{H}^{\beta-1} \otimes \overset{\infty}{\mathfrak{H}}$ into $\otimes\mathfrak{H}^{\alpha-1} \otimes \overset{\infty}{\mathfrak{H}}$ $(\alpha, \beta = 0, 1, \ldots;\ \otimes\mathfrak{H}^{-1} \otimes \overset{\infty}{\mathfrak{H}} = \dot{\mathfrak{H}};\ \otimes\mathfrak{H}^{0} \otimes \overset{\infty}{\mathfrak{H}} = \overset{\infty}{\mathfrak{H}}$; these latter conventions must be kept in mind throughout what follows). Extend this operator to an operator from $\otimes\mathfrak{H}^k$ into $\otimes\mathfrak{H}^j$ $(j \geq \alpha,\ k \geq \beta)$ as follows. Embed the space $\otimes\mathfrak{H}^\alpha$ in $\otimes\mathfrak{H}^j$ $(j \geq \alpha)$, identifying it with $\otimes\mathfrak{H}^\alpha \otimes e \otimes \cdots \otimes e$. Consider the operator of orthogonal projection of $\otimes\mathfrak{H}^k$ onto $\otimes\mathfrak{H}^{\beta-1} \otimes \overset{\infty}{\mathfrak{H}}$; it is equal to $1 \otimes \cdots \otimes 1 \otimes \overset{\infty}{P} \otimes \dot{P} \otimes \cdots \otimes \dot{P}$, where $\dot{P}$ and $\overset{\infty}{P}$ are the projectors in $\mathfrak{H}$ onto $\dot{\mathfrak{H}}$ and $\overset{\infty}{\mathfrak{H}}$. The required extension is defined by the formula $A_{\alpha\lambda}(1 \otimes \cdots \otimes 1 \otimes \overset{\infty}{P} \otimes \dot{P} \otimes \cdots \otimes \dot{P})$.

Theorem. In order that the matrix $K = (K_{jk})_{j,k=0}^{\infty}$ be a moment matrix, it is necessary and sufficient that it admit a representation in the form of the following operator integral, convergent in norm:

$$K_{jk} = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{k} \int_{-\infty}^{\infty} \lambda^{j+k} \, dT_{\alpha\beta}(\lambda)(1 \otimes \cdots \otimes 1 \otimes \overset{\infty}{P} \otimes \dot{P} \otimes \cdots \otimes \dot{P}) \qquad (j, k = 0, 1, \ldots), \tag{4}$$

where $T_{\alpha\beta}(\Delta)$ is an arbitrary bounded operator measure acting from $\otimes\mathfrak{H}^{\beta-1} \otimes \overset{\infty}{\mathfrak{H}}$ into $\otimes\mathfrak{H}^{\alpha-1} \otimes \overset{\infty}{\mathfrak{H}}$, for which the matrix $(T_{\alpha\beta}(\Delta))_{\alpha,\beta=0}^{\infty}$ is p.o.

In (4) one may, of course, bring the sum under the integral sign and include it in the differential; however, the resulting differential is less convenient to describe than $dT_{\alpha\beta}(\lambda)$.

We obtain the proof of the representation in the following way. The matrix K generates a generalized kernel $\mathcal{K}$ associated with a chain of positive and negative spaces $H_- = H_0 \supset H_+$, where

$$H_0 = \bigoplus_{j=0}^{\infty} \otimes \mathfrak{H}^j,$$

and H_+ is chosen in the proper way. Conditions (1) and (3) ensure that the general theorem on integral representation of p.o. kernels (([3], Chapter VIII, § 1)) is applicable to such a kernel, leading to the equality

$$\mathcal{K} = \int_{-\infty}^{\infty} \Omega_\lambda \, d\rho(\lambda)$$

(cf. (4)). The elementary kernels $\Omega_\lambda = (\Omega_{\lambda;jk})_{j,k=0}^{\infty}$ will, for each of the indices j, k, be eigenfunctions for certain equations of difference type with left shift, adjoint to the shift (2) (see (5)). Finding the general solution of these equations and substituting it into the expression for $\mathcal{K}$, one can arrive at (4). The finding of the general solution is based on the following lemma:

Lemma. Let the sequence $f = (f_0, f_1, ...)$, $f_j \in \otimes\mathfrak{H}^j$, where $\mathfrak{H}$ is some Hilbert space, satisfy the equation

$$(1 \otimes \cdots \otimes 1 \otimes \dot{P}) f_{j+1} = \lambda f_j \qquad (j = 0, 1, ...). \tag{5}$$

Here $\dot{P} f_1 = (f_1, e) \in C_1$, $e \in \mathfrak{H}$ is fixed, $\|e\| = 1$. The general solution of equation (5) has the form

$$f_j = \sum_{\alpha=0}^{j} \lambda^{j-\alpha} f_\alpha^\infty \otimes e \otimes \cdots \otimes e \qquad (j = 0, 1, ...),$$

where $f_\alpha \in \otimes\mathfrak{H}^{\alpha-1} \otimes \mathfrak{H}^\infty$ $(\alpha = 0, 1, ...)$ are arbitrarily prescribed "initial data" (equal to $(\underbrace{1 \otimes \cdots \otimes 1}_{\alpha-1} \otimes \overset{\infty}{P}) f_\alpha$).

2^0. Suppose that the representation (4) holds. We apply (4) to g_k, multiply scalarly by f_j, and sum over j, k. As a result we obtain Parseval' s equality

$$\langle f, g \rangle = \sum_{\alpha,\beta=0}^{\infty} \int_{-\infty}^{\infty} ((dT_{\alpha\beta}(\lambda)) \tilde{f}_\beta(\lambda), \tilde{g}_\alpha(\lambda)), \tag{6}$$

where $f, g \in \Phi(\mathfrak{H})$, and the "Fourier transform" $\tilde{f}(\lambda) = (\tilde{f}_0(\lambda), \tilde{f}_1(\lambda), ...)$ of the vector f is defined by the equality

$$\tilde{f}_\alpha(\lambda) = \sum_{j=\alpha}^{\infty} \lambda^j (\underbrace{1 \otimes \cdots \otimes 1}_{\alpha-1} \otimes \overset{\infty}{P} \otimes \underbrace{\dot{P} \otimes \cdots \otimes \dot{P}}_{j-\alpha}) f_j \in \otimes\mathfrak{H}^{\alpha-1} \otimes \mathfrak{H}^\infty$$

$$(\alpha = 0, 1, ...). \quad (7)$$

The following inversion formula is valid for the transform (7):

$$f_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}(\tilde{f}_0(\lambda) + \tilde{f}_1(\lambda) + \cdots)\right)(0) \quad (n = 0, 1, ...).$$

It follows easily from (7) that $(\widetilde{Tf})_\alpha(\lambda) = \lambda f_\alpha(\lambda)$ $(\alpha = 0, 1, ...)$; therefore, using (6), we conclude that $\langle Tf, g\rangle = \langle f, Tg\rangle$ $(f, g \in \Phi(\mathfrak{H}))$, i.e., (4) implies (3).

It is not difficult to show that the totality $\widetilde{\Phi}(\mathfrak{H})$ of Fourier transforms consists of all finite sequences $(P_0(\lambda), P_1(\lambda), ...)$, where $P_\alpha(\lambda)$ is a vector polynomial of the form $\lambda^\alpha(a_{\alpha,1} + \lambda a_{\alpha,2} + \cdots + \lambda^n a_{\alpha,n})$, $a_{\alpha,j} \in \otimes\mathfrak{H}^{\alpha-1} \otimes \mathfrak{H}^\infty$. Completion of $\widetilde{\Phi}(\mathfrak{H})$ with respect to the scalar product (6) leads to a Hilbert space $\widetilde{H}_k$, isometric to H_k, the completion of $\Phi(\mathfrak{H})$ with respect to $\langle\cdot,\cdot\rangle$. The shift (2) on $\Phi(\mathfrak{H})$ generates in H_k a Hermitian operator A, isometric to the operator of multiplication by λ, defined on $\widetilde{\Phi}(\mathfrak{H})$ in $\widetilde{H}_k$. The operator measure $dT_{\alpha\beta}(\lambda)$ in (4) is determined uniquely by K if and only if the closure of A is self-adjoint in H_k. The following sufficient criterion for self-adjointness holds: if $\|K_{jk}\| \le p_{jp}k$ $(j, k = 0, 1, ...)$, and the class $C(p_n)$ is quasi-analytic, then the closure of A is self-adjoint in H_k. The proof of this assertion is carried out similarly to the proof of Theorem 5.8, Ch. VIII of ([3]) and is essentially set forth in ([4]); the literature on the question is presented in the same work. We shall not dwell on the transfer of further results of the power moment problem to our problem.

3^0. We now give some constructions more special than in item 1^0. Suppose there is a chain of positive and negative spaces $\otimes\mathfrak{H}_-^j \supseteq \otimes\mathfrak{H}_0^j \supseteq \otimes\mathfrak{H}_+^j$ $(j = 0, 1, ...)$; in each such chain its own involution $^- = {}^-_j$ is introduced (for the terminology and notation see ([3]), Ch. I, §§ 1-2 or ([5]); we furnish scalar products and norms in $\otimes\mathfrak{H}_-^j$, $\otimes\mathfrak{H}_0^j$, and $\otimes\mathfrak{H}_+^j$ with the indices $-, 0, +$, respectively). Consider the matrix $K = (K_{jk})_{j,k=0}^\infty$, where $K_{jk} \in \otimes\mathfrak{H}_-^{j+k}$ and the analogous positive-definiteness condition (1) is satisfied:

$$\sum_{j,k=0}^\infty (K_{jk}, u_j \otimes \overline{u}_k)_0 \ge 0 \quad (u = (u_0, u_1, ...) \in \Phi(\mathfrak{H}_+)). \quad (8)$$

Let $e \in \mathfrak{H}_+$, $\|e\|_+ = 1$, and let K be such that, similarly to (3),

$$\sum_{j,k=0}^\infty (K_{j+1,k}, v_j \otimes e \otimes \overline{u}_k)_0 = \sum_{j,k=0}^\infty (K_{j,k+1}, v_j \otimes \overline{u}_k \otimes e)_0 \quad (u, v \in \Phi(\mathfrak{H}_+)). \quad (9)$$

We associate with K the matrix of operators $K = (K_{jk})_{j,k=0}^{\infty}$, where K_{jk} acts continuously from $\otimes\mathfrak{H}_+^k$ into $\otimes\mathfrak{H}_+^j$ according to the equality

$$(K_{jk}u_k, v_j)_+ = (K_{jk}, v_j \otimes \overline{u}_k)_0$$

(the role of $\mathfrak{H}$ is played by $\mathfrak{H}_+$). By theorem, item 1^0, K_{jk}, and hence also K_{jk}, admits the representation (4). This representation can be written in such a form that, instead of the operators $T_{\alpha\beta}(\Delta)$, elements of tensor products occur. For this purpose consider a Hilbert space $\mathfrak{H}_{++} \subseteq \mathfrak{H}_+$ such that the embedding $\mathfrak{H}_{++} \to \mathfrak{H}_+$ is quasi-nuclear, and construct chains

$$\otimes\mathfrak{H}_{--}^j \supseteq \otimes\mathfrak{H}_{-}^j \supseteq \otimes\mathfrak{H}_0^j \supseteq \otimes\mathfrak{H}_+^j \supseteq \otimes\mathfrak{H}_{++}^j \qquad (j = 0, 1, ...).$$

In addition, we shall suppose that the involution $- = -j$ is continued to the last chain and has the property that

$$\overline{\varphi_j \otimes \psi_k^{j+k}} = \overline{\psi}_k \otimes \overline{\varphi}_j \quad (\varphi_j \in \otimes\mathfrak{H}_{--}^j,\ \psi_k \in \otimes\mathfrak{H}_{--}^k;\ j, k = 0, 1, ...). \tag{10}$$

By virtue of the theorem on the kernel, for arbitrary $u_\beta \in \otimes\mathfrak{H}_{++}^\beta$, $v_\alpha \in \otimes\mathfrak{H}_{++}^\alpha$,

$$(T_{\alpha 3}(\Delta)(\underbrace{1 \otimes \cdots \otimes 1}_{\beta-1} \otimes \overset{\infty}{P})u_\beta,\ v_\alpha)_+ = (T_{\alpha 3}(\Delta),\ v_\alpha \otimes \overline{u}_\beta)_0,$$

where

$$T_{\alpha 3}(\Delta) \in \otimes\mathfrak{H}_{--}^{\alpha+\beta}.$$

We shall embed $\otimes\mathfrak{H}_{--}^{\alpha+\beta}$ in $\otimes\mathfrak{H}_{--}^{j+k}$ $(j \geq \alpha,\ k \geq \beta)$, identifying this space with

$$\otimes\mathfrak{H}_{--}^\alpha \otimes \underbrace{\varepsilon \otimes \cdots \otimes \varepsilon}_{j-\alpha} \otimes \underbrace{\overline{\varepsilon} \otimes \cdots \otimes \overline{\varepsilon}}_{k-\beta} \otimes\mathfrak{H}_{--}^\beta \subseteq \otimes\mathfrak{H}_{--}^{j+k};$$

the element corresponding to $\xi_{\alpha\beta}$ will be denoted by

$$\xi_\alpha\langle \otimes \underbrace{\varepsilon \otimes \cdots \otimes \varepsilon}_{j-\alpha} \otimes \underbrace{\overline{\varepsilon} \otimes \cdots \otimes \overline{\varepsilon}}_{k-\beta} \rangle_\beta.$$

Here $\varepsilon \in \mathfrak{H}_- \subseteq \mathfrak{H}_{--}$ is determined from the equality

$$(\varepsilon, u)_0 = (e, u)_+ \qquad (u \in \mathfrak{H}_+).$$

It is not hard to see that now representation (4) can be rewritten in the form

$$K_{jk} = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{k} \int_{-\infty}^{\infty} \lambda^{j+k}\, dT_\alpha\langle \otimes\varepsilon \otimes \cdots \otimes \varepsilon \otimes \overline{\varepsilon} \otimes \cdots \otimes \overline{\varepsilon} \rangle_\beta(\lambda)$$

$$(j, k = 0, 1, ...). \tag{11}$$

4^0. A functional of Wightman type will be called a sequence

$$W = (W_0, W_1, \dots),$$

where $W_j \in \otimes \mathfrak{H}_-^j$, for which the matrix

$$(K_{jk})_{j,k=0}^{\infty} = (W_{j+k})_{j,k=0}^{\infty}$$

is positive definite, i.e., (8) is fulfilled. Since from (8) it always follows that $K_{jk} = \overline{K}_{kj}$, W is real:

$$W_j = \overline{W}_j \qquad (j = 0, 1, \dots).$$

Now condition (9), thanks to (10), is fulfilled automatically for any $e = \overline{e}$. Conversely, it is easy to see that if (9) holds for any $e = \overline{e} \in \mathfrak{H}_+$, $\|e\|_+ = 1$, then K_{jk} depends on the sum:

$$K_{jk} = W_{j+k} \qquad (j, k = 0, 1, \dots).$$

Thus, representation (11) holds for any $e = \overline{e} \in \mathfrak{H}_+$, $\|e\|_+ = 1$, if and only if

$$(K_{jk})_{j,k=0}^{\infty} = (W_{j+k})_{j,k=0}^{\infty},$$

where W is a functional of Wightman type. From what was said in item 2^0 it is easy to see that, if the class

$$C(\sqrt{\|W_{2n}\|_-})$$

is quasi-analytic, then the measure in each representation (11) for W_{j+k} is determined uniquely.

In conclusion let us note that in items $3^0, 4^0$ one can pass in the usual way from elements of $\otimes \mathfrak{H}_{--}^j$ to considerations of functionals over nuclear linear topological spaces. In particular, such spaces may be spaces of functions of j variables $x_1, \dots, x_j \in (-\infty, \infty)$; as the involution $-j$, satisfying (10), one may take the passage

$$f(x_1, x_2, \dots, x_{j-1}, x_j) \to \overline{f(x_j, x_{j-1}, \dots, x_2, x_1)},$$

where the bar denotes complex conjugation.

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Received
14 III 1966

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Note: Figure translations are in progress. See original paper for figures.