The asymptotic behavior of solutions of a class of Volterra' s integro-differential equations

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Abstract

A system of the form

$$x'(t) = z(x(t), t) - \int_0^{tK} (t, s)\psi(x(s), s)\, ds, \qquad (*)$$

is considered, where $x(t)$, $z(x, t)$, $\psi(x, t)$ are n-dimensional vectors; $K(t, s)$ is a symmetric matrix of the n-th order, and $\psi_i(x, t)$ depends only on x_i and t. Conditions are specified under which the solution of the system (*) is bounded and when $\lim_{t\to\infty} x^{(j)}(t) = 0$ $(j = 0, 1, 2)$. Similar results are obtained for the system

$$x'(t) = z(x(t), t) - \int_{t-L}^{tK} (t, s)\psi(x(s), s)\, ds.$$

Bibliography: 3 items.

Full Text

Introduction

This work investigates the stability and asymptotic behavior of solutions to a class of nonlinear integro-differential equations. We consider systems of the form:

$$x'(t) = f(t) - b(t) - A(t)\phi(x(t), t) - \int_0^t K(t, s)\psi(x(s), s)\, ds \qquad (1.1)$$

where $x(t)$ and $f(t)$ are n-dimensional vectors, and $A(t)$ and $K(t, s)$ are $n \times n$ matrices. This formulation generalizes several models previously studied in the literature [**?**, **?**], specifically addressing cases where the kernel $K(t, s)$ and the nonlinearities ϕ and ψ satisfy certain monotonicity and positivity conditions.

1.1 Fundamental Assumptions and Definitions

We assume that the functions $f(t), b(t), \phi(x,t), \psi(x,t), A(t)$, and $K(t,s)$ are continuous for $0 \le s \le t < \infty$. Let D_1 and D_2 denote the partial derivatives with respect to the first and second arguments of the kernel, respectively, such that $D_1 K(t,s) = \frac{\partial}{\partial t} K(t,s)$ and $D_2 K(t,s) = \frac{\partial}{\partial s} K(t,s)$. We define $D = D_1 + D_2$. The following conditions are imposed on the system components:

- The matrix $A(t)$ is positive definite, satisfying $(A(t)x, x) > 0$ for $x \neq 0$, and there exists a constant M such that $\|A(t)\| \le M$.
- The kernel $K(t,s)$ satisfies $K(t,0) > 0$, $D_1 K(t,0) < 0$, and $D_2 K(t,s) \ge 0$. Furthermore, we assume the mixed derivative condition $D_1 D_2 K(t,s) \le 0$.
- The nonlinear functions $\phi(x,t)$ and $\psi(x,t)$ are such that $(\phi(x,t), \psi(x,t)) > 0$ for $x \neq 0$. We also require that $\psi(x,t)$ is the gradient of some scalar potential $\Psi(x,t)$, i.e., $\psi(x,t) = \nabla_x \Psi(x,t)$, where $\Psi(x,t) = \int_0^x \psi(\xi,t)\, d\xi$.

1.2 Stability Analysis via Lyapunov Functionals

To analyze the stability of the equilibrium solution $x(t) = 0$, we construct a Lyapunov functional $V(t)$. Let $F(t) = \int_0^t \|f(s)\|\, ds$. We define the functional as:

$$V(t) = [1 + E(t)] \exp(-K_1 F(t)) \tag{1.3}$$

where $E(t)$ is a quadratic-like form involving the integral of the nonlinearity:

$$E(t) = \int_0^x \psi(\xi,t)\, d\xi + \frac{1}{2} \left(K(t,0) \int_0^t \psi(x(s),s)\, ds, \int_0^t \psi(x(s),s)\, ds \right) + \ldots \tag{1.2}$$

By differentiating $V(t)$ along the trajectories of (1.1), we obtain:

$$\begin{aligned} V'(t) = &- K_1 \|f(t)\| V(t) + \exp(-K_1 F(t)) \Big[(f(t), \psi(x(t),t)) \\ &- (A(t)\phi(x(t),t), \psi(x(t),t)) + \ldots \Big] \end{aligned} \tag{1.4}$$

Under the prescribed conditions, specifically $(A(t)\phi(x,t), \psi(x,t)) > 0$ and the monotonicity of the kernel, it can be shown that $V'(t) \le 0$. This implies the boundedness of the solution $x(t)$ for all $t \ge 0$.

2. Asymptotic Behavior

We further investigate the conditions under which the solution $x(t)$ tends to zero as $t \to \infty$. Suppose that the kernel $K(t,s)$ is of the convolution type $K(t-s)$. If the integral $\int_0^\infty \|K(t)\|\, dt$ is bounded and the external forcing $f(t)$ vanishes asymptotically, we can apply Barbalat' s Lemma.

2.1 Linearized System and Resolvents

Consider the linear operator associated with (1.1). Let $W(t,s)$ be the resolvent kernel satisfying:

$$\frac{\partial}{\partial t}W(t,s) + A(t)W(t,s) + \int_s^t M(t,\tau)W(\tau,s)\,d\tau = 0 \tag{2.6}$$

where $M(t,s) = -D_2L(t,s)$ and $L(t,s) = A(s) + \int_s^t K(\tau,s)\,d\tau$. If the matrix $A(0) + \int_0^\infty K(s,0)\,ds$ is positive definite, the solution can be expressed via the variation of constants formula:

$$u(t) = \left[W(t,0) + \int_0^t W(t,s)F(s)\,ds\right]Bx_0 \tag{2.10}$$

where B is a constant matrix. As $t \to \infty$, if $W(t,s) \to 0$, then the state $x(t)$ converges to the equilibrium.

3. Systems with Time Delays

The analysis extends to systems with a constant time delay $L > 0$:

$$x'(t) = z(x(t),t) - \int_{t-L}^t K(t,s)\psi(x(s),s)\,ds \tag{4.1}$$

For such systems, the initial condition is defined on the interval $[-L,0]$. We assume the kernel $K(t,s)$ vanishes for $s = t - L$. By constructing a modified Lyapunov functional:

$$V(t) = (1 + E(t))\exp(-K_1F(t)) \tag{4.6}$$

where $E(t)$ accounts for the history of the state $\psi(x(s),s)$ over the interval $[t-L,t]$, we can establish similar stability criteria. Specifically, if the mixed derivatives of the kernel satisfy $D_1D_2K(t,s) \leq 0$ and the potential function $\Psi(x,t)$ is positive definite, the solution $x(t)$ and its derivative $x'(t)$ both approach zero as $t \to \infty$.

Conclusion

The results presented here provide sufficient conditions for the global stability and asymptotic convergence of nonlinear integro-differential systems. These conditions rely on the passivity of the nonlinear components and the monotonicity of the integral kernel, consistent with previous findings in [**?**, **?**, **?**]. The use of exponential weighting in the Lyapunov functional allows for the inclusion of non-vanishing perturbations $f(t)$, provided they are integrable.

Figures

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DIFFERENTIAL EQUATIONS

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ON THE SINGULAR SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION WITH A DEVIATING ARGUMENT OF NEUTRAL TYPE

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Let P be some operator. The solution $x(t, \lambda)$ of the equation $x(t) = P(x(t), \lambda)$ is called *singular* if $x(t, \lambda) \to \infty$ as $\lambda \to 0$.

Many mathematicians have been concerned with the question of the existence of singular solutions. For example, A. A. Temlyakov [1], H. N. Nazarov [2], M. M. Smirnov [3], P. P. Rybin [4], Dzh. G. Yusif-zade [5] have dealt – with singular solutions of integral equations, and the works of K. T. Akhmedova [6] [7, 8] are devoted to singular solutions of integro-differential equations.

In our work, we investigate the question of the existence of a singular solution of a nonlinear integro-differential equation with a deviating argument of neutral type in the case where the integrands are polynomials with respect to the desired function. For simplicity of exposition, let us consider an equation of the particular form

$$\dot{x}(t) = \lambda \int_0^1 [A_1(t, s)x(s-\tau) + A_2(t, s)\dot{x}(s-\tau) + A_3(t, s)x^2(s)]\,ds + $$
$$+ \lambda^2 \int_0^1 [B_1(t, s)x(s) + B_2(t, s)x(s-\tau) + B_3(t, s)x(s-\tau)\dot{x}(s-\tau)]\,ds, \quad (1)$$

where $x(s) = \frac{dx}{ds}$; $0 < \tau < 1$ — constant deviation; λ — parameter; $A_2(t, s)$, $B_r(t, s)$ — continuous functions of their arguments in the domain $D\{0 \leqslant t, s \leqslant 1\}$.

We will seek the solution $x(t, \lambda)$ of equation (1) for $0 \leqslant t \leqslant 1$ in the class C of continuous functions having bounded derivatives, under the initial condition

$$x(t, \lambda) = \varphi(t, \lambda) \text{ an } E_0 = [-\tau, 0], \quad (2)$$

where $\varphi(t, \lambda)$ is continuous and has a bounded derivative.

Assume that the initial function $\varphi(t, \lambda)$ is representable as a series

$$\varphi(t, \lambda) = \frac{\lambda_0}{\lambda}\varphi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k \varphi_k(t), \quad (3)$$

where $\lambda_0 \neq 0$ is a yet unknown number, which will be determined later.

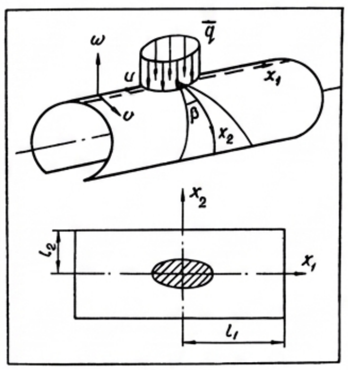
We will seek the solution to problem (1), (2) in the form of a series

$$x(t, \lambda) = \frac{\lambda_0}{\lambda}\psi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k \psi_k(t). \quad (4)$$

Figure 1: Figure 1

Further in the work [2] a generalization is made to the case of an arbitrary characteristic of the thread (dependence of the force in it on deformation) and the equilibrium equations of the net shell with extensible threads are obtained.

Using the example of the equations for a cylindrical shell, we show that the introduction of extensibility of threads significantly simplifies the study of the correctness of the formulation of the boundary value problem (the concept of correctness, as is known, means the existence and uniqueness of at least a generalized solution in some sense continuously dependent on the right-hand side of the system). The system of equilibrium equations for a cylindrical shell turns out to be a system of strong–elliptic type, for which one can use the results obtained in [3]. For such systems of equations, it becomes also possible to apply approximate difference methods developed in [5 and 6], and also a wide class of variational methods [7].



Pig. 1.

Particular attention is paid to obtaining the weakest possible restrictions on the size of the domain, under which the correctness of the studied boundary value problem is guaranteed (these restrictions will further be called sufficient conditions for correctness). Results of calculations are presented, showing the influence of parameters included in the coeftificients of the system, namely the angle between threads β and the paramete μ, charakterizzing the extensibility of the threads, on the eeronents, on the sufficient conditions for

correctness. Atre same time, one should keep in mind that although the conditions obtained in the work on the size of the domains, apparently, can be refined, but anyway in the most important for most practice range of variation of β and μ [$0<\mu<0,05$; $45°<\beta<60°$] they are sufficient to guarantees nue the correctness of the formulated boundary value problems and the possibility of applying approximate methods [6, 7] for the case of symmetric loads.

The equations describing small deformations of a cylindrical net shell (Fig. 1), obtained in [2], can be written in vector form as follows: This eocquations treated trom ideas as:

$$L\bar{u}(x) \equiv R\bar{u}(x) + \mu Q\bar{u}(x) = \bar{f}(x). \tag{1}$$

Here $(x) = (x_1, x_2)$ — dimensionless coordinates on the surface of the shell; $\bar{u}(x_1, x_2)$ — the vector of total displacement to be determined,

$$\bar{v}(x_1, x_2) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \\ w(x_1, x_2) \end{pmatrix};$$

Figure 2: Figure 2

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$\bar{f} = \mu i\bar{q}$, $\bar{q}$ — a vector proportional to the external lagd; L, R, Q — square matrices of differential operators:

$$R = s^2 \begin{pmatrix} t^2D_1^2 + D_2^2 & 2D_1D_2 & D_1 \\ 2D_1D_2 & D_1^2 + \frac{1}{t^2} D_2^2 & \frac{1}{t^2} D_2 \\ -D_1 & -\frac{1}{t^2} D_2 & -\frac{1}{t^2} \end{pmatrix};$$

$$Q = \begin{pmatrix} s^2D_1^2 + c^2D_2^2 & -2s^2D_1D_2 & -(1+s^2)D_1 \\ -2s^2D_1D_2 & s^2t^2D_1^2 + s^2D_2^2 & s^2D_2 \\ (1+s^2)D_1 & -s^2D_2 & t^2D_1^2 + D_2^2 + c^2 \end{pmatrix},$$

where β — the argle between the threads of two families $\left(0<\beta<\frac{\pi}{2}\right)$; $c=\cos\beta$; $s=\sin\beta$; $t=\tan\beta$; $\mu=\frac{N_0}{E_k}$; N_0 — the initial force in the sholl from any-internal pressure; E_k — the modulus material tesd.

Brounary value problem for system (1) uenion, that $\bar{f}(x)$ is a neriodicatic funoktion in x_1:

$$\bar{f}(x_1 + 2l_1, x_2) = \bar{f}(x_1, x_2),$$

is formulated as follows: in the poloce

$$\Pi = \{-\infty < x_1 < +\infty,\ -l_2 \leqslant x_2 \leqslant l_2,\ l_2 > 0\}$$

find a pelιation $\bar{u}$ of custem (1), ydoblethropping the ycnosbin periodivnoctin no x_1

$$\bar{u}(x_1 + 2l_1, x_2) = \bar{u}(x_1, x_2) \tag{2}$$

and the nero granuary condition in x_2

$$\bar{u}|_{x_2=\pm l_2} = 0. \tag{3}$$

By dirtee of condition (2), one on erpætrict to finding the peliation nostended solvew in the obmain

$$\bar{\Omega} = \{x;\ -l_1 \leqslant x_1 \leqslant l_1;\ -l_2 \leqslant x_2 \leqslant l_2\}.$$

Let us bredume the npoctpanctbo W_2^r vectortsir fynction $\bar{u}(x)$, onpedenened and conperuious tmether with their npousesitives of form

$$D^\alpha\bar{u} \equiv D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2}\bar{u} \quad (\alpha = (\alpha_1, \alpha_2),\quad |\alpha| = \alpha_1 + \alpha_2 \leqslant r)$$

in obnacth $\bar{\Omega}$, ydobletroperting yclorious (2) and

$$D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} u(-l_1 + 0, x_2) = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} u(l_1 - 0, x_2) \quad |\alpha| \leqslant r. \tag{4}$$

Hormy in W_2^r onpedenezs clediyom an obraow:

$$\|\bar{u}\|_{W_2^r} = \left\{ \sum_{|\alpha|=r} [D^\alpha u, D^\alpha u] \right\}^{1/2},$$

Figure 3: Figure 3

$$U_g(X,\ \bar{Y},\ \lambda) = Y - Q\Big[2 + 2\lambda + |\lambda_0|a +$$

$$+ \lambda(X+Y) + \frac{\lambda(X+Y+X^2) + \lambda^2(\lambda X + XY)}{|\lambda_0|a}\Big] = 0, \tag{10}$$

where

$$M > \{\sup_D |A_t(t,\ s)|;\ \sup_D |B_t(t,\ s)|\}, \qquad Q = |\lambda_0| Ma(1 + |\lambda_0| R).$$

The functions U_1 and U_2 at the point . . . uvall

$$X = a_o = Q\left[\frac{|\varphi_0(0)|}{|\lambda_0| Ma} + 2 + |\lambda_0|a\right],$$

$$Y = b_o = Q(2 + |\lambda_0|a), \quad \lambda = 0$$

vanish, and the poink, the determinant $\dfrac{\partial(U_1,\ U_2)}{\partial(X,\ Y)}$ at this point is different from zero. Therefore, on the basis of the theorem on the existence of an implicit function, system (10) has, relative to X and Y, a unique, holomorphic in the parameter λ, solution in the neighborhood of $\lambda = 0$:

$$X = \sum_{k=0}^{\infty} a_k k^k, \qquad Y = \sum_{k=0}^{\infty} b_k \lambda^k. \tag{11}$$

It is not difficult to prove that series (11) majorize respectively series (9) and, consequently, series (9) converge absolutely and uniformly at least in the region in which series (11) converge.

We formulate the obtained result in the form of a theorem.

Theorem 1. *If the problem* (6_{-1}) *has a nontrivial solution, bhiiangling to the class C, and* λ_0 *is not a characteristic number of the integral equation* (8), *then the problem* (1), (2) *has a unique singular solution, bhiiangiing to the class C and representable in the form of series* (4).

2. Let now λ_0 be a characteristic number of equation (8). For the sake of simplicity, we restrict ourselves to the consideration of the case when λ_0 is a characteristic number of the first rank.

Let us denote by $w(t)$ the eigenfunction of the kernel $\Phi(t,\ s)$, and by $v(t)$ the eigenfunction of the adjoint kernel, corresponding to the value λ_0. In this case, for the solvability of equation (7_0), it is necessary and sufficient to fulfill the condition

$$\int_0^t \Big\{\varphi_0(0) + \lambda_0 \int_0^t\int_0^1 [A_1(s,\ s)\psi_{-1}(s-\tau) + A_2(z,\ s)\psi_{-1}(s-\tau) +$$

$$+ \lambda_2^2 B_2(z,\ s)\psi_{-1}(s-\tau)\psi_{-1}(s-\tau)]\, ds dz\Big\}\, v(t)\, dt = 0. \tag{12}$$

When condition (12) is fulfilled, equation (7_0) has the solution

$$\varphi_0(t) = C_0 w(t) + u_0(t),$$

where C_0 is an arbitrary constant; $u_0(t)$ is a particular solution of equation (7_0).

For the solvability of equation (7_1), it is necessary and sufficient to fulfill the condition

$$P_0 C_0^2 + Q_0 C_0 + T_0 = 0. \tag{13}$$

Figure 4: Figure 4

Proof. We have:

$$-[Q\bar{u},\ \bar{u}] = -s^2[(D_1^2y, u) + r^2(D^2u, u) + 2(D_1D_0z, u) +$$
$$+ (D_1w, u) + 2(D_1D_1z, v) + \frac{1}{t^2}(D^2o, v) +$$
$$+ (D^2o, v) + \frac{1}{t^2}(D_2w, v) - (D_1u, w) -$$
$$- \frac{1}{t^2}(D_1v, w) - \frac{1}{t^2}(w, w)\}. \qquad (9)$$

Since for any y and z from W_2^1 integration by parts yields (D_1y, z) $z) = -(y, D_1z)$, $(D_2y, z) = -(y, D_2z)$, then, applying these equalities for moving one of the derivatives of the first factor in some terms of the left side of (9) to the second factor, it is not difficult to obtain

$$-[Q\bar{u},\ \bar{u}] = s^2[\|D_1y\|^2 + \|tD_1u\|^2 + 2(D_1u, D_1v) +$$
$$+ 2(D_1u, w) + 2(D_1u, D_2v) + \left\|\frac{1}{t}D_2o\right\|^2 + \|D_1v\|^2 +$$
$$+ 2\left(\frac{1}{t}D_2v, \frac{1}{t}w\right) + \left\|\frac{1}{t}w\right\|^2,$$

from which the validity of the lemma follows.

Lemma 2. *For anebing $\bar{u}$ in W_2^2 the following equality holds*

$$-[Q\bar{u},\ \bar{u}] = -s^2(J_1 + J_2) + J, \qquad (10)$$

where J_1 and J_2 are defined by formulas (8), and

$$J = \sum_{i=1}^{n}(\|tD_1u_i\|^2 + \|D_gu_i\|^2) - 2(w, D_1u) + 2(w, D_gv). \qquad (11)$$

Proof. Writing out $-[Q\bar{u},\ \bar{u}]$ and again applying integration by by-parts, we find

$$-[Q\bar{u},\ \bar{u}] = \|cD_du\|^2 + \|sD_1v|^2 + \|sD_2v|^2 +$$
$$+ \|stD_1v|^2 + \|D_2w\|^2 + \|tD_2w|^2 -$$
$$- \|cw\|^2 - 2s^3(D_1u, D_2v) - 2s^2(D_1v, D_gu) =$$
$$- 2(D_1u, w) - 2s^2(D_1u, w) + 2(1 - c^3)(D_2v, w) =$$
$$= -s^2(J_1 + J_2) + \|D_gu_i\|^2 + t^2\|D_1v\|^2 + t^2\|D_1v\|^2 +$$
$$+ \|D_2v\|^2 + \|D_2w\|^2 + \|tD_yw\|^2 - 2(D_1u, w) +$$
$$+ 2(D_1v, w) = -s^3(J_1 + J_2) + J.$$

Figure 5: Figure 5

$$+ \dot{\psi}_{-1}(s-\tau)\, u_1(s-\tau)\Big) + \psi_0(s-\tau)\, \dot{\psi}_0(s-\tau)\Big]\Big\}\, v(t)\, ds\, dz\, dt +$$

$$+ \frac{1}{\lambda_0} \int_0^1 \frac{v(s)\, \psi_0(s)\, u_1(s)}{\psi_{-1}(s)}\, ds$$

— are known numbers debecting ot C_{0j}.

Assuming $Q_1 \neq 0$, let us find two values of C_{1i} $(i = 1,\ 2)$ for C_1, with $C_{11} \neq C_{12}$ by virttye of the assumption $C_{01} \neq C_{02}$. Then

$$\psi_{1i}(t) = C_{1i}\omega(t) + u_1(t) \qquad (i = 1,\ 2).$$

When lefolned the yellotion (15) ypanation (7_2) byd hwet the pelietion

$$\psi_2(t) = C_2\omega(t) + u_2(t),$$

where C_2 — a new proursolary constoant; $u_2(t)$ — a vacthor peliation of ypanution (7_3). The yelobion polvebinoction ypauretion (7_3)

$$Q_1C_2 + T_2 = 0$$

let us find C_{2i} $(i = 1,\ 2)$. Dpodoiwing this ppocess, the proursolary constoant whress C_n, entayring the pelection $\psi_n(t) = C_n\omega(t) + u_n(t)$ ypauretion (7_n), hadnrin its yclobion polvebimotion

$$Q_1C_n + T_n = 0$$

of ypanition (7_{n+1}), it

$$T_n = \varphi_{n+1}(0) \int_0^1 v(t)\, dt + \int_0^1 \int_0^t \int_0^1 \Big\{A_1(z,\ s)\, u_n(s-\tau) + A_2(z,\ s)\, \dot{u}_n(s-\tau) +$$

$$+ B_1(z,\ s)\, \psi_{n-1}(s) + B_2(z,\ s)\, u_{n-1}(s-\tau) + B_2(z,\ s)\, \times$$

$$\times \Big[\lambda_n\Big(\psi_{-1}(s-\tau)\, \dot{u}_n(s-\tau) + \dot{\psi}_{-1}(s-\tau)\, u_n(s-\tau)\Big) +$$

$$+ \sum_{i=0}^{n-1} \psi_i(s-\tau)\, \dot{\psi}_{n-i-1}(s-\tau)\Big]\Big\}\, v(t)\, ds\, dz\, dt +$$

$$+ \frac{1}{2\lambda_0} \int_0^1 \frac{v(s)\Big[2u_n(s)\, \psi_0(s) + \sum_{i=1}^{n-1} \psi_i(s)\, \psi_{n-i}(s)\Big]}{\psi_{-1}(s)}\, ds.$$

Thus, the coefficients of the desired series are determined successively and belong to class C.

Uniform convergence of the series

$$\sum_{k=0}^{\infty} \lambda^k\, \psi_{ki}(t) \qquad (i = 1,\ 2)$$

and their first derivatives with respect to t can be proved by the method of majorants.

Now let us consider the case, when $Q_1 = 0$. In this case the solution to problem (1), (2) can be sought in the form of a series

$$x(t,\ \lambda) = \frac{\lambda_0}{\lambda}\, \psi_{-1}(t) + \frac{1}{\lambda^{\frac{1}{2}}}\, \psi_0(t) + \sum_{k=1}^{\infty} \lambda^{\frac{k-1}{2}}\, \psi_k(t). \qquad (16)$$

Figure 6: Figure 6

$$R_\omega = \left[\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} - \frac{(1-q-r)^2}{t^2 - \varepsilon} \right] \|\omega\|^2. \quad (15)$$

Thus, the following is proved

Lemma 4. For any $\bar{u} \in \overline{W}_2^1$, the inequality holds

$$-[L\bar{u}, \bar{u}] \geqslant \mu \{\varepsilon \|\bar{u}\|^2_{\overline{W}_2^1} + R_u + R_v + R_\omega\}, \quad (16)$$

where $\varepsilon > 0$ is an arbitrary number from the interval (0, min (1, t^2)), and R_u, R_v, R_ω are defined by formulas (13)—(15).

It remains to note the conditions for the velocity of non-negativities

$$R_u \geqslant 0, \quad R_v \geqslant 0, \quad R_\omega \geqslant 0,$$

and then those conditions, (5) will be satisfied, when, with condition (3)

$$\|\bar{u}\|^2_{\overline{W}_2^1} \geqslant \varepsilon_1 \|\bar{u}\|^2_{\overline{W}_2^1}, \quad \varepsilon_1 > 0.$$

Necessary conditions are nonnegativities of R_u, R_v, R_ω and promissory u, v, w of the considered class of function by get:

$$1 - \varepsilon + t^2 p = \frac{t^1 p^2 \mu}{s^2 (1-\mu) + t^2 p \mu} > 0, \quad (17)$$

$$t^2 (1-p) - \varepsilon > 0, \quad 0 < \varepsilon < 1. \quad (18)$$

$$\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} = \frac{(1-q-r)^2}{t^2 - \varepsilon} > 0. \quad (19)$$

Let us assume for now, that these conditions related. Then it is seen, that q downs to bate > 0.

We new use the fact, that for any function z, immeeting two continuous properties to x_2 and pavatiemeeted at hyls with $x_2 = \pm l_2$, the following inequality holds

$$\int_{-l_2}^{l_2} (D_x z)^2 dx_2 > \frac{1}{4\gamma^2} \int_{-l_2}^{l_2} z^2 dx_2, \quad \gamma^2 = \frac{l_2^2}{\pi^2}.$$

If in this case z (x_2) is an even function, to

$$\int_{-l_3}^{l_3} (D_x z)^2 dx_2 > \frac{1}{\gamma^2} \int_{-l_2}^{l_3} z^2 dx_2.$$

Let $\omega = 2$, if z (x_2) is newethest to x_2, and in dpyrux cases $\omega = 1$. Then soctationnable conditions for nonnegativity R_u, R_v, R_ω will be (17)—(19) and

$$\left(1 - \varepsilon + t^2 p = \frac{t^2 p^2 \mu}{s^3 (1-\mu) + t^2 p \mu}\right) \frac{1}{4\gamma^2} - \frac{r^2}{r^2 - \varepsilon} > 0, \quad (20)$$

Figure 7: Figure 7

$$(1-\varepsilon)\frac{\omega^2}{4\gamma^2}-\frac{\left(1+\frac{q}{t^2}\right)^2}{1-\varepsilon}\geqslant 0. \tag{21}$$

Since we are interested in the maximum value of γ for which (20), (21) will be valid when (17)—(19) are fulfilled for any sufficiently small $\mu\geqslant 0$, it is clear that p must be taken as large as possible.

Therefore, we set $p=1-\frac{\varepsilon}{t^2}$. Then (18) is fulfilled as an equality, and (17) transforms into

$$\rho(t, \varepsilon, \mu)=1+t^2-2e-\frac{\mu(t^2-s)^2}{(1-\mu)s^2+\mu(t^2-\varepsilon)}>0. \tag{22}$$

Instead of (20), (21), we easily obtain the equivalent inequalities

$$r^2\gamma^2\leqslant\frac{t^2-\varepsilon}{4}\rho(t, \varepsilon, \mu), \tag{23}$$

$$\gamma\leqslant\frac{(1-\varepsilon)\omega t^2}{2(t^2+q)}. \tag{24}$$

Obviously,

$$\rho(t, \varepsilon, \mu)>1+t^2-2e-(t^2-\varepsilon)=1-\varepsilon$$

and (17), (22) will be fulfilled, since $\varepsilon<1$.

Let us denote the set (ε, q, r) for which $q>0$ and (19) are fulfilled by E_t, and let $\gamma(\varepsilon, q, r)$ denote the supremum of the values of γ satisfying (23), (24) for fixed ε, q, r from E_t.

Let us find

$$\gamma(t)=\sup_{(\varepsilon, q, r)\in E_t}\gamma(\varepsilon, q, r)^{*}).$$

It is easy to verify that

$$\gamma(t)=\sup_{(0, q, r)\in E_t}\gamma(0, q, r).$$

From (19), (23), (24) with $\varepsilon=0$, we obtain

$$(1+\alpha)q^2-2(2-r)q+(1-r)^2\leqslant 0, \tag{25}$$

$$r^2\gamma^2\leqslant\frac{t^2}{4}\rho(\beta, \mu), \tag{26}$$

$$\gamma\leqslant\frac{\omega t^2}{2(t^2+q)}, \tag{27}$$

where

$$\alpha=\frac{\mu}{s^2(1-\mu)};$$

$$\rho(\beta, \mu)=1+t^2-\frac{\mu t^t}{(1-\mu)s^2+\mu t^2}>1.$$

* The dependence of γ on μ is omitted in some cases for brevity.

Figure 8: Figure 8

It is also useful to note that in case 2), of course, it will be

$$b = 1 - \frac{1}{a} > 0.$$

Thus, for fixed μ and β, $\gamma(t, \mu)$ can be easily calcuclated using the formylas indicated above.

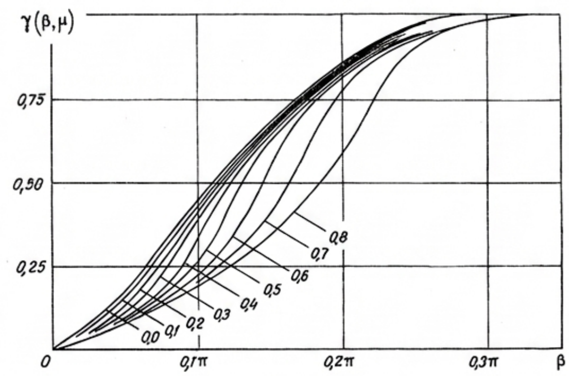

Fig. 2.

Now it is not difficult to verify that the following holds true

Theorem 1. *If* $\mu < 1$ *and* $\frac{I_2}{\pi} < \gamma(t, \mu)$, ... $\gamma(t,$ *is determined by formulas* (30), (32), *then there exists a sufficiently small* $\delta > 0$, *such that for any functions u* ... $\bar{W}_2^2$ *the inequality holds*

$$-[L\bar{u}, u] \geqslant \mu\delta\|u\|^2_{w_2^1}.$$

In conclusion, we present several graphs of $\gamma(t, \mu)$ for fixed μ (Fig. 2), from which it can be concluded that for $\mu > 0$ there is some deterioration of the sufficient conditions for correctness compared to the graph of $\gamma(t, 0)$, but it should be noted that, firstly, this deterioration refers to the case of those $\mu > 0$ that have very little practical significance, and, – secondly, this is a deterioration of only the sufficient conditions obtained by the technique adopted in the work, which, generally speaking, can be refined.

We also note the following: although the system of equations for dn–etermining displacements and forces in threads for the case $\mu = 0$, given in [1], cannot be obtained from system (1) by substituting $\mu = 0$, the graph of $\gamma(t, 0)$ gives sufficient conditions on the dimensions of the region, guaranteeing the uniqueness of the solution of the boundary value problem for system [1] and in the case of inextensible threads.

Figure 9: Figure 9

Indeed, if system [1] is written in the form

$$\boldsymbol{P}\bar{u} = \begin{pmatrix} p_1\bar{u} \\ p_2\bar{u} \\ \cdots\cdots \\ p_3\bar{u} \end{pmatrix} = \boldsymbol{f},$$

where $\bar{u}$ is an unknown five-dimensional vector, three components of which are displacements, and the rest are forces; $\boldsymbol{P}$ is a matrix differential operator; $\boldsymbol{f}$ is a known five-dimensional vector, then, calculating $[P\bar{u}, \bar{u}]_0 = \sum_{i=1}^{5} (p_i, \bar{u}, \bar{u})$, one can verify that the forces in the threads will drop out of the functional $[P\bar{u}, \bar{u}]_0$, and this functional itself will have a form exactly coinciding with that obtained from the right side of formula (12) after substituting $\mu = 0$.

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Figure 10: Figure 10

§ 3. Representation of vectors E and H via Macdonald integrals

Before writing down the main result of this paragraph, let us introduce the following notations:

$$\mathbf{r} = \rho(\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}) + z\,\mathbf{k} = \\ = z\,\mathbf{e}_1 + \rho\left[\cos(\varphi-\psi)\,\mathbf{e}_2 + \sin(\varphi-\psi)\,\mathbf{e}_3\right], \tag{3.1}$$

$$\mathbf{r}_0(\alpha) = \rho_0(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}) + z_0\,\mathbf{k} = \\ = z_0\,\mathbf{e}_1 + \rho_0\left[\cos(\alpha-\varphi)\,\mathbf{e}_2 + \sin(\alpha-\varphi)\,\mathbf{e}_3\right] \tag{3.2}$$

($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ — unit vectors along the axes x, y, z); note that, according to (1.8), it holds

$$R(\alpha-\varphi) = |\mathbf{r} - \mathbf{r}_0(\alpha)|. \tag{3.3}$$

The main result of this paragraph is as follows.

Theorem 2. Let $\Phi_1(\beta, \delta)$ — vector function, defined by the expression

$$\Phi_1(\beta, \delta) = \Bigg\{\left[\mathbf{e}_1, \left[\mathbf{e}_{\left(\frac{\varphi-\delta}{2}+\beta\right)}, \mathbf{e}_1\right]\right] - \\ - \frac{a(\beta, -\delta) - a(\beta, \pi+\varphi)}{2\cos\frac{\varphi+\delta}{2}}\Bigg\} \times \\ \times \frac{Pik^2}{2\sqrt{\rho\rho_0}} H_0^{(1)}(kR_0) - \Bigg\{\frac{1}{2}\left[\mathbf{e}_1, \left[\mathbf{e}_{\left(\frac{\varphi-\delta}{2}+\beta\right)}, \mathbf{e}_1\right]\right] - \\ - \frac{a(\beta, -\delta) - a(\beta, \pi+\varphi)}{\cos\frac{\varphi+\delta}{2}} + \\ + \frac{2\rho\rho_0}{R^2(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}\left(\mathbf{e}_{\beta-\delta} - 3a(\beta, -\delta)\right)\Bigg\} \times \\ \times \frac{Pik}{2\sqrt{\rho\rho_0}\,R_0} H^{(1)}(kR_0) + \\ + \frac{Pik^2}{2R(\varphi+\delta)}\left(\mathbf{e}_{\beta-\delta} - 3a(\beta, -\delta)\right) \times \\ \times M_0\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+8}{2}, kR(\varphi+\delta)\right) + \\ + \frac{Pik^3}{2}\left(\mathbf{e}_{\beta-\delta} - a(\beta, -\delta) - \frac{\mathbf{e}_{\beta-\delta} - 3a(\beta, -\delta)}{k^2R^2(\varphi+\delta)}\right) \times \\ \times M_1\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+8}{2}, kR(\varphi+\delta)\right), \tag{3.4}$$

where

$$a(\beta, \delta) = \frac{(\mathbf{r} - \mathbf{r}_n(\delta))\,(\mathbf{e}_{\beta+\delta}, \mathbf{r} - \mathbf{r}_o(\delta))}{R^3(\varphi-\delta)}. \tag{3.5}$$

Figure 11: Figure 11